NEIGHBORHOOD COMPLEXES OF STABLE KNESER GRAPHS

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Abstract. It is shown that the neighborhood complexes of a family of vertex critical subgraphs of Kneser graphs – the stable Kneser graphs introduced by L. Schrijver – are spheres up to homotopy. Furthermore, it is shown that the neighborhood complexes of a subclass of the stable Kneser graphs contain the boundaries of associahedra (simplicial complexes encoding triangulations of a polygon) as a strong deformation retract.

1. Introduction

In 1955 Martin Kneser [Kne55] conjectured that if one splits the n-subsets of a $(2n+k)$-element set into $k+1$ classes, then one of the classes contains two disjoint n-subsets. In 1978 László Lovász [Lov78] proved this conjecture – in graph-theoretic language a question about the chromatic number of the Kneser graphs – by introducing the concept of the neigborhood complex of a graph. He applied the Borsuk-Ulam theorem to show that if the neigborhood complex of a graph is topologically $(k-1)$-connected, then the graph is not $(k+1)$-colorable. Furthermore, he showed that the neighborhood complex of the Kneser graph $KG_{n,k}$ is $(k-1)$-connected, and thus proved the Kneser conjecture.

Shortly afterwards Imre Bárány [Bár78] provided a very elegant and short proof also applying the Borsuk-Ulam theorem. In the same year Alexander Schrijver [Sch78] used Bárány's method to obtain a family of vertex critical subgraphs – the family of stable Kneser graphs $SG_{n,k}$. For the definition, see Section 2.

It is natural to ask whether the chromatic number of these subgraphs $SG_{n,k}$ can be obtained by the Lovász method. If so, one would hope for a simple structure for their neighborhood complexes. They would have to be $(k-1)$-connected and not more. The most natural example of such a space is the sphere of dimension k. In fact, it was conjectured by Lovász and Schrijver [LS93] that the neighborhood complex of $SG_{n,k}$ has the homotopy type of the k-sphere. The main purpose of this paper is to show that this is indeed the case.

An interesting property of the neighborhood complexes of the family of stable Kneser graphs $SG_{2,k}$ is that they contain (as a deformation retract) the simplicial complex encoding triangulations of a $(k+4)$-gon – this is the boundary complex of a certain polytope known as the associahedron. We show this in the last section.

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2. Preliminaries

We begin by recalling the definition of a Kneser graph, its vertex critical subgraphs defined by Schrijver, and the notion of the neighborhood complex of a graph. Let $n \geq 1$ and $k \geq 0$.

▷ The vertices of the *Kneser graph* $KG_{n,k}$ are given by the n-subsets of $[2n+k] := \{1, \ldots, 2n+k\}$; two of them are joined by an edge iff they are disjoint, see Figure 1. In 1955 Kneser [Kne55] asked if the chromatic number of $KG_{n,k}$ is $k+2$.

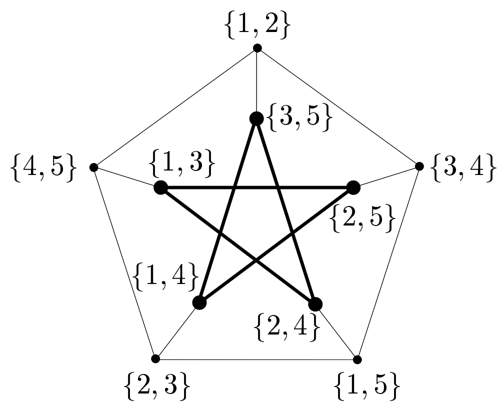

Figure 1. The Kneser graph $KG_{2,1}$ and the stable Kneser graph $SG_{2,1}$.

▷ The *neighborhood complex of a graph* $G = (V, E)$ is the simplicial complex on the vertex set V whose simplices are given by sets of vertices that have a common neighbor.

▷ A subset $v \subseteq [2n+k]$ is *quasistable* if for all $i \in [2n+k-1]$ the set $\{i, i+1\}$ is not contained in v. The set v is *stable* if it is quasistable and does not contain the set $\{1, 2n+k\}$; i.e., a subset is stable if it does not contain two neighbors in the cyclic ordering of $[2n+k]$.

▷ The vertices of the *stable Kneser graph* $SG_{n,k}$ (introduced in 1978 by A. Schrijver) are the stable n-subsets of $[2n+k]$; two of them are joined by an edge iff they are disjoint. $SG_{n,k}$ is an induced subgraph of $KG_{n,k}$, and with respect to the chromatic number it is vertex critical [Sch78]. In Figure 1 the stable Kneser graph $SG_{2,1}$ is indicated by the bold vertices and edges.

▷ The neighborhood complex of $SG_{n,k}$ is hence given by

$$\Sigma_{n,k} = \{\{v_1, \ldots, v_s\} : \forall i (v_i \text{ vertex of } SG_{n,k}), \exists \text{ vertex } v (\forall i (v \cap v_i = \emptyset))\},$$

i.e., the faces of $\Sigma_{n,k}$ are given by any family of stable n-sets v_i in the complement of a stable n-set v.

▷ For each stable n-set $v \subseteq [2n+k]$ – a vertex of $SG_{n,k}$ – we define the *neighbor facet* $\Delta_v^{n,k}$ *of* v:

$$\Delta_v^{n,k} = \{w : w \subseteq [2n+k] \text{ stable } n\text{-set}, v \cap w = \emptyset\}.$$

We will omit the superscript whenever it will not cause confusion. The neighbor facets constitute the facets of $\Sigma_{n,k}$. We thus obtain the following description of $\Sigma_{n,k}$:

$$\Sigma_{n,k} = \{F \subseteq \Delta_v : v \subseteq [2n+k] \text{ stable } n\text{-set}\}.$$

Examples.

- ▷ The only stable n-sets of $[2n+0]$ are $\{1,3,5,\ldots,2n-1\}$ and $\{2,4,6,\ldots,2n\}$ and they are complementary. Hence $\Sigma_{n,0}$ is homeomorphic to the 0-sphere $\mathbb{S}^0$.
- ▷ Each stable n-set of $[2n+1]$ is characterized by a pair $\{i,i+1\} \pmod{2n+1}$ of free positions in its complement. Thus, each such set admits two stable n-sets in its complement, constituting a 1-simplex of $\Sigma_{n,1}$. Together these 1-simplices yield a 1-sphere, and thus $\Sigma_{n,1}$ is homeomorphic to $\mathbb{S}^1$, cf. Figure 1.
- ▷ Stable 1-sets are just 1-sets by definition. Hence, for $\Sigma_{1,k}$ we obtain

$$\{\{v_1,\ldots,v_s\} : \forall i(v_i \subseteq [k+2] \text{ 1-set }), \exists \text{ 1-set } v \subseteq [k+2](\forall i(v \cap v_i = \emptyset))\}.$$

This complex is the boundary of a $(k+1)$-dimensional simplex. Therefore $\Sigma_{1,k}$ is homeomorphic to the k-sphere $\mathbb{S}^k$.

In general we cannot expect $\Sigma_{n,k}$ to be homeomorphic to the k-sphere: for example $\Sigma_{2,2}$ is a pure 3-dimensional complex. Figure 2 illustrates the two prototypes $\Delta_{\{1,3\}}$ and $\Delta_{\{1,4\}}$ of facets of $\Sigma_{2,2}$.

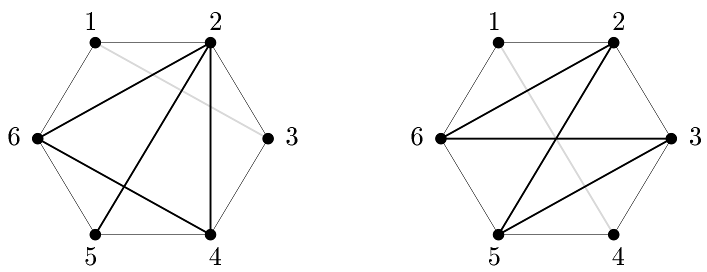

FIGURE 2. The black diagonal edges in each polygon yield the vertices of a 3-simplex in $\Sigma_{2,2}$.

Moreover, the complexes $\Sigma_{n,k}$ are in general not even pure, as can be seen by considering the two prototypes of facets of $\Sigma_{3,2}$.

We now review a few helpful lemmas that are needed for the proofs. If a topological space X is covered by a family $(X_\alpha)_{\alpha\in A}$ of subspaces we use the notation $X_\sigma = \bigcap_{\alpha\in\sigma} X_\alpha$ for subsets $\sigma \subseteq A$.

Gluing Lemma ([tD71])**.** Let $f\colon X \to Y$ be a continuous map of topological spaces, $(X_\alpha)_{\alpha\in A}$ and $(Y_\alpha)_{\alpha\in A}$ be closed, finite-dimensional, locally finite coverings of X and Y, respectively. Assume that the inclusions $X_\sigma \hookrightarrow X_\tau$ respectively $Y_\sigma \hookrightarrow Y_\tau$ are cofibrations for finite subsets $\tau \subseteq \sigma \subseteq A$. If $f(X_\alpha) \subseteq Y_\alpha$ for all $\alpha \in A$ and $f|_{X_\sigma}\colon X_\sigma \to Y_\sigma$ is a homotopy equivalence for all finite $\sigma \subseteq A$, then f itself is a homotopy equivalence.

In a contractible space the inclusion of a subspace can always be extended to a map of the cone over that subspace. Furthermore, this extended map is a homotopy equivalence. From this we deduce the following consequence of the Gluing Lemma.

Corollary. Let X be a simplicial (or CW-) complex and A and B contractible subcomplexes with $A \cup B = X$. Then X is homotopy equivalent to the suspension $\operatorname{susp}(A \cap B)$.

Multicone Lemma ([BW99], [For99]). Let $\Gamma_1 \subseteq \cdots \subseteq \Gamma_l = \Gamma$ be simplicial complexes, and let $\Gamma_0 = \emptyset$. Assume there exist vertices $w_1, \dots, w_l$ such that for $i = 1, \dots, l$ the assignment

$$F \longmapsto \begin{cases} F \cup \{w_i\} & \text{, if } w_i \notin F, \\ F \setminus \{w_i\} & \text{, if } w_i \in F. \end{cases}$$

maps $\Gamma_i \setminus \Gamma_{i-1}$ into itself. Then Γ is collapsible.

Nerve Lemma (see e.g. [Bjö94] or [McC67]). Let Γ be a simplicial complex and $(\Gamma_i)_{i \in I}$ a family of subcomplexes such that $\Gamma = \bigcup_{i \in I} \Gamma_i$ and every finite nonempty intersection $\Gamma_{i_1} \cap \cdots \cap \Gamma_{i_s}$ is contractible. Then the *nerve complex*

$$\mathcal{N}(\Gamma_i) := \left\{ \sigma \subseteq I : \sigma \text{ finite}, \bigcap_{i \in \sigma} \Gamma_i \neq \emptyset \right\}$$

is homotopy equivalent to Γ.

Contraction Lemma (see e.g. [Bre93, Chapter VII]). If $f \colon A \longrightarrow X$ is a cofibration and A is contractible, then the collapse $X \longrightarrow X/A$ is a homotopy equivalence.

3. The neighborhood complexes of stable Kneser graphs are homotopy spheres

In this section we prove our main result.

Theorem 3.1. *The simplicial complex $\Sigma_{n,k}$ is homotopy equivalent to the k-sphere $\mathbb{S}^k$ for all $n \geq 1$ and $k \geq 0$.*

The proof of the Theorem, which proceeds by induction on k, is given at the end of this section. We will cover the complex $\Sigma_{n,k}$ by two contractible subcomplexes that are shown to intersect up to homotopy in $\Sigma_{n,k-1}$.

Let for all $n, k \geq 1$ the subcomplexes $A_{n,k}$ and $B_{n,k}$ of $\Sigma_{n,k}$ be defined by:

$$A_{n,k} = \{F \subseteq \Delta_v : v \text{ vertex of } \Sigma_{n,k} \text{ such that } 1 \notin v\}$$
$$B_{n,k} = \{F \subseteq \Delta_v : v \text{ vertex of } \Sigma_{n,k} \text{ such that } 1 \in v\}.$$

Obviously the union of $A_{n,k}$ and $B_{n,k}$ is $\Sigma_{n,k}$.

Proposition 3.2. For all $n, k \geq 1$ the complexes $A_{n,k}$ and $B_{n,k}$ are contractible to a point.

Proof. For notational reasons we prove that

$$A'_{n,k} = \{F \subseteq \Delta_v : v \text{ vertex of } \Sigma_{n,k} \text{ such that } 2n+k \notin v\}$$

is contractible. In fact, we show that it is collapsible using the Multicone Lemma.

In order to use this Lemma we begin by defining a sequence $\Gamma_1 \subseteq \cdots \subseteq \Gamma_l = A'_{n,k}$ of ascending subcomplexes of $A'_{n,k}$. To do so order all stable n-sets $v \subseteq [2n+k]$, $2n+k \notin v$ lexicographically. Say $v_1 \prec \cdots \prec v_l$. For $i = 1, \dots, l$ define

$$\Gamma_i = \{F \subseteq \Delta_{v_j} : 1 \leq j \leq i\}.$$

Next we define a set $w_1, \dots, w_l$ of vertices. Consider $v_i = \{a_1, \dots, a_n\} \subseteq [2n+k-1]$, and define the stable n-set $w_i = \{a_1+1, \dots, a_n+1\} \subseteq [2n+k]$, $i = 1, \dots, l$. Note that $w_i \in \Delta_{v_i}$ for $i = 1, \dots, l$. See Figure 3.

The last step is to investigate the map defined in the Multicone Lemma. Let $i \in \{1, \dots, l\}$ and $F \in \Gamma_i \setminus \Gamma_{i-1}$ a simplex. If $w_i \notin F$ we map F to $F \cup \{w_i\}$. It is easy to see that $F \cup \{w_i\} \in \Gamma_i \setminus \Gamma_{i-1}$. If $w_i \in F$ then we map F to $F \setminus \{w_i\}$. In this case $F \setminus \{w_i\} \in \Gamma_i \setminus \Gamma_{i-1}$ for the following reason. Consider the *support* $\operatorname{supp}(F) = \bigcup F \subseteq [2n+k]$ of F. The fact that $F \in \Gamma_i \setminus \Gamma_{i-1}$ implies that the lexicographically smallest stable n-set in $[2n+k] \setminus \operatorname{supp}(F)$ is v_i. Furthermore, $w_i \in F$ implies that in fact the first n elements of $[2n+k] \setminus \operatorname{supp}(F)$ are given by

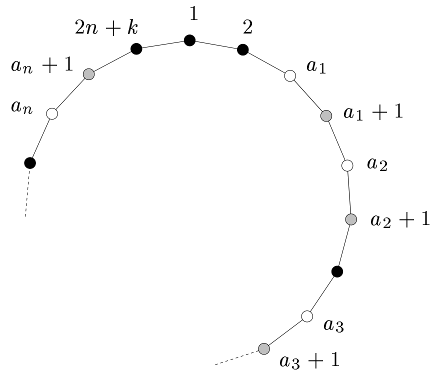

FIGURE 3. The vertices v_i and w_i.

the set v_i. Hence $F \setminus \{w_i\} \in \Gamma_{i-1}$ only if the set $\{a_1, a_1+1, a_2, a_2+1, \ldots, a_n, a_n+1\}$ contains a stable n-set that precedes v_i in the lexicographic order. But this is not the case. Thus the Multicone Lemma applies.

The complex

$$\begin{aligned} B_{1,k} &= \{F \subseteq \Delta_v^{1,k} : v \subseteq [2+k] \text{ 1-set}, 1 \in v\} \\ &= \{F \subseteq \Delta_{\{1\}}^{1,k}\} \end{aligned}$$

is a k-dimensional simplex and therefore contractible. Thus, to prove the second part of Proposition 3.2 it suffices to show that $B_{n,k} \simeq B_{n-1,k}$ for all $n \geq 2$.

Consider the covering $(2^{\Delta_w})_{w \in \{v \subseteq [2n+k] : v \text{ stable } n\text{-set}, 1 \in v\}}$ of $B_{n,k}$, where 2^{Δ_w} is an abbreviation for the complex $\{F : F \subseteq \Delta_w\}$ of all faces of Δ_w. By the Nerve Lemma we obtain

$$\begin{aligned} B_{n,k} \simeq \mathcal{N}(2^{\Delta_w}) &= \big\{\{v_1, \ldots, v_s\} : v_i \text{ stable}, 1 \in v_i, 2^{\Delta_{v_1}} \cap \cdots \cap 2^{\Delta_{v_s}} \neq \emptyset\big\} \\ &= \{\{v_1, \ldots, v_s\} : v_i \text{ stable}, 1 \in v_i, \exists \text{ stable } v(\forall i(v \cap v_i = \emptyset))\} \\ &= \{\{v_1, \ldots, v_s\} : v_i \text{ stable}, 1 \in v_i, \\ &\qquad \exists \text{ stable } v(2 \in v, 2n+k \in v\, \forall i(v \cap v_i = \emptyset))\}, \end{aligned}$$

where the last equation follows by the stability of the vertices. Hence by deleting the element $1 \in [2n+k]$ and identifying $2, 2n+k \in [2n+k]$ (see Figure 4) we obtain the following identification:

$$\begin{aligned} \mathcal{N}(2^{\Delta_w}) &\cong \big\{F \subseteq \Delta_v^{n-1,k} : v \subseteq [2(n-1)+k] \text{ stable } (n-1)\text{-set}, 1 \in v\big\} \\ &= B_{n-1,k}. \end{aligned}$$

□

In order to get a good description of the intersection $A_{n,k+1} \cap B_{n,k+1}$ we need the following preliminary fact.

Lemma 3.3. Let $v_w, v_b \subseteq [2n+k+1]$ be two stable n-subsets such that $1 \notin v_w$ and $1 \in v_b$. Then there exists a stable n-subset $v \subseteq [2n+k+1]$ with the following properties:

(i) $v \subseteq v_w \cup v_b$,
(ii) $1 \notin v$, and
(iii) $2 \notin v$ or $2n+k+1 \notin v$.

Proof. Call $i \in [2n+k+1]$ *black* if $i \in v_b$ and *white* if $i \in v_w$. In general, it can happen that i is black and white.
Case (1): $2 \notin v_w$ **or** $2n+k+1 \notin v_w$. Set $v = v_w$.

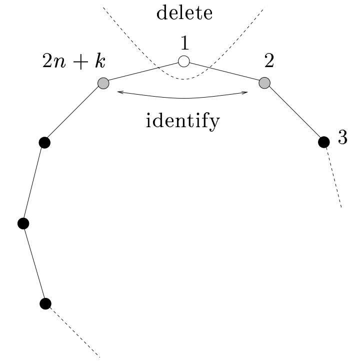

FIGURE 4. Deletion and identification of elements in $[2n+k]$.

Case (2): $2 \in v_w$ and $2n+k+1 \in v_w$. Consider the sequence $2, 3, 4, \ldots$. By the stability of v_b and v_w the numbers in the sequence are colored white and black alternately until there is a non-colored number. Non-colored numbers exist since $2n+k+1 > 2n$. Let i be the smallest number such that $i+1$ is not colored (see Figure 5).

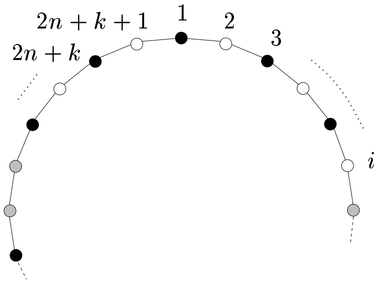

FIGURE 5. Points in $[2n+k+1]$ colored alternately.

If i is white, then set $v = \{2, 4, \ldots, i\} \cup \{\text{all black numbers } > i\}$, and if i is black, then set $v = \{3, 5, \ldots, i\} \cup \{\text{all white numbers } > i\}$. □

Proposition 3.4. For all $n \geq 1$ and $k \geq 0$ we have a homotopy equivalence $A_{n,k+1} \cap B_{n,k+1} \simeq \Sigma_{n,k}$.

That is, $A_{n,k+1}$ and $B_{n,k+1}$ intersect up to homotopy in the neighborhood complex of a stable Kneser graph of dimension one less.

Proof. We compute the intersection

$$\begin{aligned} &A_{n,k+1} \cap B_{n,k+1} = \\ &= \big\{F \subseteq \Delta^{n,k+1}_{v_w} \cap \Delta^{n,k+1}_{v_b} : v_w, v_b \subseteq [2n+k+1] \text{ stable } n\text{-sets}, 1 \notin v_w, 1 \in v_b\big\} \\ &= \{\{v_1, \ldots, v_s\} : \forall i (v_i \subseteq [2n+k+1] \text{ stable } n\text{-set}, 1 \notin v_i), \exists \text{ stable } n\text{-set } v \\ &\qquad \text{such that } 1 \notin v \text{ and } (2 \notin v \text{ or } 2n+k+1 \notin v), \forall i (v \cap v_i = \emptyset)\}, \end{aligned}$$

where the last equation is justified by Lemma 3.3. Now we delete the number $1 \in [2n+k+1]$, since it is not used for the vertices in the intersection. This forces us to consider quasistable n-sets as vertices.

$$\begin{aligned} A_{n,k+1} \cap B_{n,k+1} \cong \{\{v_1, \ldots, v_s\} &: \forall i (v_i \subseteq [2n+k] \text{ quasistable } n\text{-set}), \\ &\exists \text{ stable } n\text{-set } v \subseteq [2n+k], \forall i (v \cap v_i = \emptyset)\} \\ = \{F \subseteq \bar{\Delta}_v^{n,k} &: v \subseteq [2n+k] \text{ stable } n\text{-set}\}, \end{aligned}$$

where $\bar{\Delta}_v^{n,k} := \{w : w \subseteq [2n+k] \text{ quasistable } n\text{-set}, v \cap w = \emptyset\}$. Denote by $I_{n,k}$ this identified intersection of $A_{n,k+1} \cap B_{n,k+1}$. We observe the following.

- ▹ $\Sigma_{n,k} \subseteq I_{n,k}$, and hence an isomorphic copy of $\Sigma_{n,k}$ is contained in $A_{n,k+1} \cap B_{n,k+1}$.
- ▹ $\{v : v \subseteq [2n+k] \text{ quasistable } n\text{-set}, 1, 2n+k \in v\}$ are the vertices of $I_{n,k}$ not used by $\Sigma_{n,k}$.

In order to describe $I_{n,k}$ in terms of $\Sigma_{n,k}$ we define two subcomplexes $C_{n,k}$ and $D_{n,k}$ which measure the surplus.

$$\begin{aligned} C_{n,k} &= \left\{F \subseteq \bar{\Delta}_v^{n,k} : v \subseteq [2n+k] \text{ stable } n\text{-set}, 1, 2n+k \notin v\right\} \\ D_{n,k} &= \left\{F \subseteq \Delta_v^{n,k} : v \subseteq [2n+k] \text{ stable } n\text{-set}, 1, 2n+k \notin v\right\}. \end{aligned}$$

The facets of $C_{n,k}$ constitute all facets of $I_{n,k}$ containing vertices of $I_{n,k}$ not in $\Sigma_{n,k}$. Hence we have

$$I_{n,k} = \Sigma_{n,k} \cup C_{n,k}.$$

The intersection of $\Sigma_{n,k}$ and $C_{n,k}$ is given by simplices of $\Sigma_{n,k}$ that are contained in a facet of $C_{n,k}$, and therefore

$$D_{n,k} = \Sigma_{n,k} \cap C_{n,k}.$$

In order to show the homotopy equivalence

$$I_{n,k} \simeq \Sigma_{n,k}$$

it suffices to prove that $C_{n,k}$ and $D_{n,k}$ are contractible. The sufficiency can be seen by using the Gluing Lemma or the Contraction Lemma.

The contractibility of $C_{n,k}$ and $D_{n,k}$ is shown by using a multicone construction argument analogous to the one that we used in the proof of Proposition 3.2 to show the contractibility of $A_{n,k}$ (based on the Multicone Lemma). □

Proof of Theorem 3.1. The Theorem can be deduced by induction from Propositions 3.2 and 3.4 using the Corollary of the Gluing Lemma. □

4. Neighborhood complexes and associahedra

In the case $n = 2$ the stable n-subsets of $[k+4]$, i.e., the vertices of $\Sigma_{2,k}$, correspond to diagonal edges of a $(k+4)$-gon. For any stable 2-set $v \subseteq [k+4]$ the simplex Δ_v contains faces that correspond to triangulations of the $(k+4)$-gon, compare Figure 6. In fact, the simplicial complex Θ_k consisting of all k-dimensional simplices in $\Sigma_{2,k}$ that correspond to triangulations is a triangulated sphere. It was shown by Haiman [Hai84] and Lee [Lee89] that this sphere arises as the boundary complex of a $(k+1)$-dimensional simplicial polytope, which is called *associahedron* for the fact that triangulations of the $(k+4)$-gon correspond to ways of parenthesizing a sequence of $k+3$ symbols. We show that the subcomplex Θ_k of $\Sigma_{2,k}$ is in fact a strong deformation retract.

Consider the covering $(2^{\Delta_v})_{v \text{ stable}}$ of $\Sigma_{2,k}$ and the induced covering $(T_v)_{v \text{ stable}}$ of Θ_k, where $T_v = \Theta_k \cap 2^{\Delta_v}$. For example in the case $k = 2$ the faces of $\Delta_{\{1,3\}}$ given by triangulations shown in Figure 6 yield the facets of $T_{\{1,3\}}$.

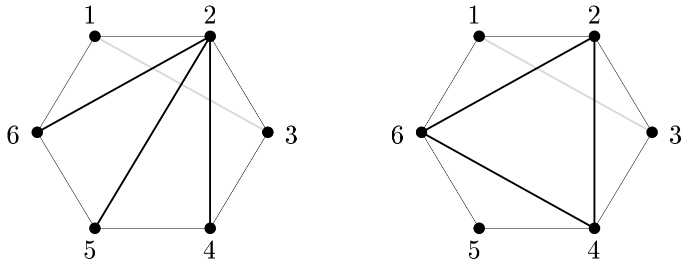

FIGURE 6. The triangulations in the facet $\Delta_{\{1,3\}}$.

Lemma 4.1. For all $\sigma \subseteq \{v : v \text{ stable 2-set of } [k+4]\}$ the following inclusion is a homotopy equivalence

$$i : \bigcap_{v\in\sigma} T_v \hookrightarrow \bigcap_{v\in\sigma} 2^{\Delta_v}.$$

Proof. It suffices to show that for all σ

- ▷ the space $\bigcap_{v\in\sigma} T_v$ is empty if and only if $\bigcap_{v\in\sigma} 2^{\Delta_v}$ is empty,
- ▷ and $\bigcap_{v\in\sigma} T_v$ is contractible in the case where it is non-empty.

The first statement is clear. The second statement follows from the fact that any non-empty space $\bigcap_{v\in\sigma} T_v$ is a cone, which can be seen as follows. Consider a maximal sequence of consecutive numbers in $\bigcup_{v\in\sigma} v \subseteq [k+4]$ modulo $k+4$. The edge given by the predecessor and successor modulo $k+4$ of this sequence is contained in every facet of $\bigcap_{v\in\sigma} T_v$ (cf. Figure 7). □

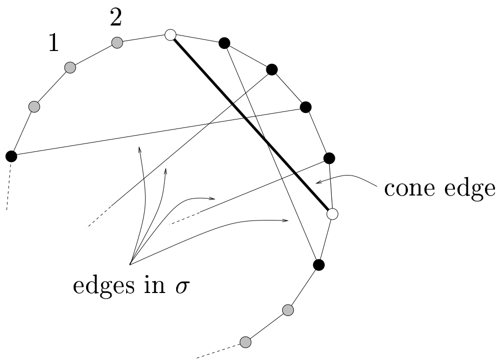

FIGURE 7. The cone peak edge.

The Gluing Lemma now tells us that the inclusion map $i\colon \Theta_k \hookrightarrow \Sigma_{2,k}$ is a homotopy equivalence, i.e., Θ_k is a weak deformation retract of $\Sigma_{2,k}$. Since $(\Sigma_{2,k}, \Theta_k)$ is a pair of simplicial complexes some elementary results from homotopy theory (cf., e.g., [Spa66, p. 31 & p. 402]) imply the following.

Theorem 4.2. *The subcomplex Θ_k (or, equivalently, the boundary complex of the $(k+1)$-dimensional associahedron) is a strong deformation retract of $\Sigma_{2,k}$.* □

Remark 4.3. Note that Theorem 4.2 implies Theorem 3.1 for the case $n = 2$. This suggests the possiblility of a more general result, based on finding suitable generalizations of the class of associahedra for $n > 2$. We ask: Is there for all $n \geq 1$ and $k \geq 0$ a $(k+1)$- dimensional simplicial polytope whose boundary complex is contained in $\Sigma_{n,k}$ as a strong deformation retract?

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