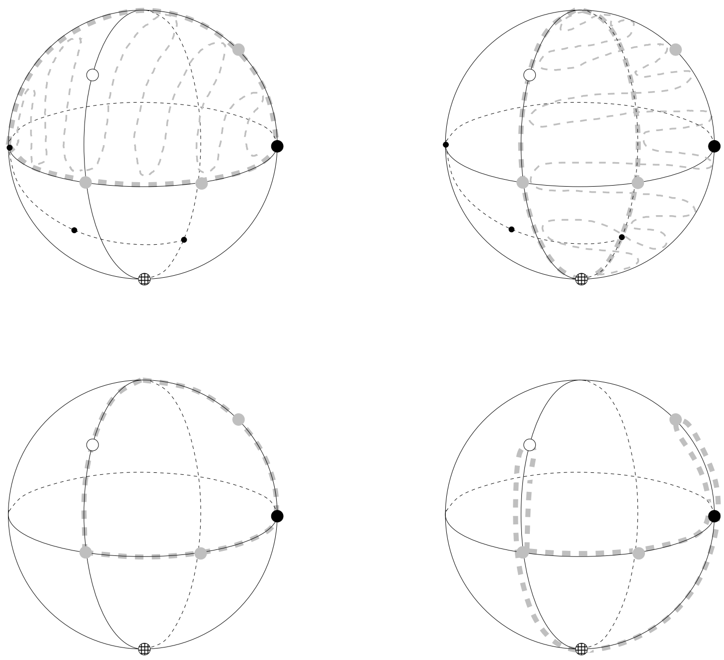
Cohomology Rings of Subspace Arrangements and the Topology of Stable Kneser Graphs

Mark de Longueville



Cohomology Rings of Subspace Arrangements and the Topology of Stable Kneser Graphs

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Promotionsausschuß:

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Acknowledgements

It happened by chance. I had just finished my diploma thesis about knot theory and did not yet know what the future would bring. Then on one of the many parties in that summer I ran into Christian Haase. We were fellow students at Freie Universität Berlin and he had written his diploma thesis some time before me. He was telling me about his new career in the Graduate School "Algorithmische Diskrete Mathematik". I was most suspicious about it, but could not help to ask him whether he thought that I could apply for it, as well. He encouraged me to do it.

And so I met with Günter M. Ziegler who was telling me about fascinating mathematics: combinatorial problems that somehow all involved topology ...

This was two and a half years ago and I can look back at a wonderful time as a member of the Graduate School "Algorithmische Diskrete Mathematik"[1]. I want to thank the School for supporting me and for providing a perfect working environment during all this time. I am grateful to Bettina Felsner who has been doing a great job coordinating the school.

Of course, I owe a lot to my advisor Günter M. Ziegler. He taught me great combinatorics and introduced me to the world of topological combinatorics, as well as to many mathematicians doing it. Without his support, encouragement and valuable advice this thesis would not have come to existence. His "Discrete Geometry" group at Technische Universität Berlin provided the very stimulating atmosphere and space for discussions on problems in very different areas of mathematics. Thank you!

Furthermore, I want to thank Volkmar Welker for introducing me to many topics in algebraic combinatorics, such as combinatorial commutative algebra and the algebraic combinatorics of lattice polytopes, and for many fruitful discussions.

I am very grateful to my coauthor Anders Björner (Chapter 3). The DAAD gave me the opportunity to visit him at KTH Stockholm in December 1998[2]. This led to a very enjoyable stay and fruitful collaboration.

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Thanks to Boris Shapiro with whom I had a very stimulating conversation on arrangements when I was in Stockholm.

Also, I want to thank my second coauthor Carsten A. Schultz (Chapter 2). After I had finished the work on coordinate subspace arrangements (Chapter 1) I gave a talk in the colloquium of the Graduate School, which led to the joyful collaboration with him.

For setting the fundament of my knowledge in algebraic topology and much more, I thank Elmar Vogt from Freie Universität Berlin, who advised my diploma thesis. His way of doing mathematics fascinated me from the beginning and I am happy that he was there when I started learning topology. Thank you for the many conversations and advice over the last couple of years!

During my studies at Freie Universität Prof. H. Kupisch and Prof. S. Koppelberg had great influence on my mathematical education. I hope that some of their influence can be found in this thesis. I am grateful to them for many wonderful courses and seminars in algebra and the foundations of mathematics.

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Berlin, February 2000 *Mark de Longueville*

Contents

Introduction

Subspace Arrangements

A *subspace arrangement* is a finite family of subspaces of euclidean space $\mathbb{R}^n$. The combinatorics and topology of complements of such arrangements are well studied objects and enjoy a long history of research.

The origins can be seen in the *combinatorial cheese cutting problem*: into how many pieces can cheese (euclidean space) be divided by a certain number of cuts (hyperplanes)? Not surprisingly, the maximal number of pieces for cheese of any dimension was determined more than a hundred years ago by the *Swiss* Ludwig Schläfli [Sch01].

The following pattern of the problem is typical for results in the field of subspace arrangements:

- ▷ some *topological property* of the arrangement: here the number of connected components of the complement

is described

- ▷ by some of its *combinatorial data:* here the dimension of the space and the number of hyperplanes.

In order to obtain the maximal number of components for the cheese cutting problem the hyperplanes have to be in general position. For the combinatorics of the general case we want to mention the work of Branko Grünbaum [Grü71] and Thomas Zaslavsky [Zas75].

Vladimir I. Arnol'd's work [Arn69] on the braid arrangement launched broad research on the description of the cohomology of the complement of *complex hyperplane arrangements.* Important ground work in this direction was done by Egbert Brieskorn [Bri73] and resulted in a combinatorial description of the cohomology ring of the complement by Peter Orlik and Louis Solomon [OS80]. Their result describes a presentation of the cohomology ring of the complement in terms of the intersection lattice and the dimension function of the arrangement.

Another interesting branch of research developed around the question which arrangements have the property that the complement is an Eilenberg-MacLane $K(\pi, 1)$-space. Pierre Deligne [Del72] has shown that a large class of arrangements in fact has this property: the class of complexified real simplicial arrangements.

Strong progress on the description of the cohomology of the complement by combinatorial data was done by Mark Goresky and Robert MacPherson applying their "Stratified Morse Theory" [GM88]. Their result completely determines the *additive structure of the cohomology* of the complement in terms of its *combinatorial data:* the intersection lattice and dimension function.

This result was strengthened by Günter M. Ziegler and Rade Živaljević [ZŽ93] by describing the homotopy type of the *link* of an arrangement in terms of its combinatorial data.

Extensive studies on the cohomology of complex hyperplane arrangements via combinatorial stratifications applying oriented matroid methods were done by Anders Björner and Günter M. Ziegler [BZ92]. Their methods were used by Eva-Maria Feichtner and Günter M. Ziegler to generalize to complex arrangements with geometric intersection lattice [FZ00].

Research on complex arrangements utilizing *rational models* for the cohomology was started by Corrado De Concini and Claudio Procesi [CP95] who presented a rational model for the cohomology ring and, in particular, showed that the ring structure depends only on the combinatorial data. Applying these techniques Sergey Yuzvinsky [Yuz98] obtained an explicit formula for the multiplication in the rational cohomology ring and derived independently a presentation for complex arrangements with geometric intersection lattice as in [FZ00] and for k-equal arrangements.

Yet another branch is leading in the direction of *commutative algebra.* A large but still special class of arrangements is given by coordinate subspace arrangements. It was shown by Vesselin Gasharov, Irena Peeva, Volkmar Welker [GPW98], Eric Babson, and Clara Chan [BC98] that the cohomology of such an arrangement relates to the Tor-algebra of the Stanley-Reisner ring associated with the combinatorics of the arrangement.

Chapter 1 of the present thesis is related to Yuzvinsky's work and has applications towards commutative algebra. It gives a complete description of the multiplication of the integral cohomology ring of a real coordinate subspace arrangement in the flavor of Yuzvinsky's description. The used methods rely on a simplicial version of the homotopy result by Ziegler and Živaljević [ZŽ93], the duality of the cross polytope and the cube, and involve elementary calculations in cubical cohomology.

Chapter 2 is motivated by a conjecture of Yuzvinsky in [Yuz98]. In a very fruitful collaboration with Carsten Schultz the ring structure of the integral cohomology of general subspace arrangements could be described. In particular, for arrangements in which all appearing codimensions are greater or equal two, we give a complete combinatorial description of the integral cohomology ring. The combinatorial data is necessarily extended to the following: intersection lattice, dimension function, and orientation information. This proves a generalization of the conjecture by Yuzvinsky mentioned above.

Stable Kneser Graphs

Laszlo Lovász's proof of the Kneser conjecture [Lov78], on the chromatic number of Kneser graphs, is an ingenious application of the Borsuk-Ulam theorem and can be considered as a prototype of a theorem in topological combinatorics.

The discovery of the vertex critical subgraphs of the Kneser graphs – the *stable Kneser graphs* – by Alexander Schrijver [Sch78] led to the question about the topology of the associated simplicial complexes defined by Lovász. The obvious guess was that these complexes should be spheres.

Together with Anders Björner this could be proved during a very enjoyable stay in Stockholm in December 1998. Our proof in **Chapter 3** is elementary and employs standard techniques from topological combinatorics.

Chapter 1

The cohomology rings of complements of coordinate subspace arrangements

1.1 Introduction and Results

This chapter is concerned with coordinate subspace arrangements, a family of (linear) subspace arrangements in real and complex space associated with simplicial complexes. For a detailed survey of subspace arrangements we refer to [Bjö94a]; all we need here is given in Section 1.2. Associated with any subspace arrangement are its link and its complement. The homology of the link, the cohomology of the complement, and in particular its ring structure, have motivated a lot of research [Arn69], [BZ92], [Bri73], [CP95], [FZ00], [GM88], [OS80], [OT92], [Zie92].

The Goresky–MacPherson formula for the homology of the link is the starting point of our investigation. By analyzing Alexander duality combinatorially in the case of coordinate subspace arrangements, we give a complete combinatorial description of the ring structure of the integral cohomology. In this analysis the duality of the cross polytope and the cube plays a crucial role.

This work was motivated by a result of S. Yuzvinsky [Yuz98] on the rational cohomology ring structure of complex arrangements. Our modeling of the cohomology of the complement was inspired by the article [BC98] of E. Babson and C. Chan.

We provide an example of a simplicial complex not containing faces of cardinality $n-1$, so that the complement of the associated real coordinate subspace arrangement is connected, that yields different ring structures for the cohomology of the complement of the associated real and complex arrangement. This answers a question by Gasharov, Peeva and Welker [GPW98].

Finally, we give an example of a coordinate subspace arrangement that yields non trivial multiplication of torsion elements.

Results

Our main result – the description of the ring structure on the cohomology of the complement C_Δ of a coordinate subspace arrangement – is based on the Goresky–MacPherson formula for the link (cf. [GM88]). After applying Alexander duality it is given in our situation by

$$\tilde{H}^i(C_\Delta;\mathbb{Z}) \cong \bigoplus_{\sigma\in\Delta} \tilde{H}_{n-i-|\sigma|-2}(\mathrm{link}_\Delta\,\sigma;\mathbb{Z}).$$

To describe the multiplication in $\tilde{H}^*(C_\Delta;\mathbb{Z})$ it suffices to describe how to multiply classes $[u]$ and $[v]$ that correspond to $[c] \in H_r(\mathrm{link}_\Delta\,\sigma;\mathbb{Z})$ and $[c'] \in H_{r'}(\mathrm{link}_\Delta\,\sigma';\mathbb{Z})$ under the Goresky–MacPherson isomorphism. Note that there is a double grading of cohomology classes by assigning the grade (r,σ) to $[u]$.
Our main result is the following.

Theorem 1.1.1. *Let $\Delta \subset 2^{[n]}$ be a simplicial complex, and let C_Δ denote the complement of the associated real coordinate subspace arrangement. The ring structure of $\tilde{H}^*(C_\Delta;\mathbb{Z})$ is given by the homomorphisms*

$$\begin{aligned} \tilde{H}_r(\mathrm{link}_\Delta\,\sigma;\mathbb{Z}) \,\otimes\, \tilde{H}_{r'}(\mathrm{link}_\Delta\,\sigma';\mathbb{Z}) &\longrightarrow \tilde{H}_{r+r'+2}(\mathrm{link}_\Delta\,\sigma\cap\sigma';\mathbb{Z}) \\ [c]\otimes[c'] &\longmapsto \begin{cases} \varepsilon\cdot[\langle i_{\sigma'}\rangle * c * c' - \langle i_\sigma\rangle * c * c'] & \textit{if } \sigma\cup\sigma' = [n], \\ 0 & \textit{otherwise.} \end{cases} \end{aligned}$$

where $i_\sigma \in [n]\setminus\sigma$ and $i_{\sigma'} \in [n]\setminus\sigma'$, and $\varepsilon \in \{\pm 1\}$ is a sign depending on n,σ,σ',r,r' computed in Section 1.3.4. If C_Δ is not connected there is additional non-trivial multiplication of cohomology classes in dimension zero.

This implies in particular that the multiplication respects the double grading of cohomology classes. The condition $\sigma\cup\sigma' = [n]$ is the "standard codimension condition" (cf., e.g., [Yuz98], [HRW99, Proposition 6]).

Corollary 1.1.2. Let $\Delta \subset 2^{[n]}$ be a simplicial complex, and let $C_\Delta^{\mathbb{C}}$ denote the complement of the associated complex coordinate subspace arrangement. The ring structure of $\tilde{H}^*(C_\Delta^{\mathbb{C}};\mathbb{Z})$ is given by the homomorphisms

$$\begin{aligned} \tilde{H}_r(\mathrm{link}_\Delta\,\sigma;\mathbb{Z}) \,\otimes\, \tilde{H}_{r'}(\mathrm{link}_\Delta\,\sigma';\mathbb{Z}) &\longrightarrow \tilde{H}_{r+r'+2}(\mathrm{link}_\Delta\,\sigma\cap\sigma';\mathbb{Z}) \\ [c]\otimes[c'] &\longmapsto \begin{cases} \varepsilon\cdot[\langle i_{\sigma'}\rangle * c * c' - \langle i_\sigma\rangle * c * c'] & \text{if } \sigma\cup\sigma' = [n], \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

where $i_\sigma \in [n]\setminus\sigma$ and $i_{\sigma'} \in [n]\setminus\sigma'$, and $\varepsilon \in \{\pm 1\}$ a sign depending on n,r,r' computed in Section 1.3.6.

The fact that the sign ε depends on σ and σ' in the real case, but not in the complex case, is the reason why in general there is no (dimension-shifting) isomorphism of graded rings between the cohomology rings of the real and complex arrangement associated with Δ (compare Corollary 1.2.2 and Section 1.4).

Example 1.1.3. There is a simplicial complex $\Delta \subset 2^{[8]}$ on eight vertices such that the following holds.

- ▷ The complement of the associated real arrangement is connected.
- ▷ The ring structure of $\tilde{H}^*(C_\Delta;\mathbb{Z})$ differs from $\tilde{H}^*(C_\Delta^{\mathbb{C}};\mathbb{Z})$.

Example 1.1.4. There is a simplicial complex $\Delta \subset 2^{[10]}$ on ten vertices such that the cohomology ring of the complement of the associated real (or complex) arrangement yields non-trivial multiplication of torsion elements.

1.2 Objects, Tools and Facts

In this section we recall basic facts on coordinate subspace arrangements, provide combinatorial models for their links and complements, and describe Lefschetz duality in the framework of cubical cohomology for the complement of a coordinate subspace arrangement.

1.2.1 Coordinate Subspace Arrangements

Simplicial complexes give rise to real and complex subspace arrangements. For that, let $\{e_1,\ldots,e_n\}$ be the standard basis of $\mathbb{R}^n$, resp. $\{e_1^{\mathbb{C}},\ldots,e_n^{\mathbb{C}}\}$ the standard basis of $\mathbb{C}^n$. Let $\Delta \subset 2^{[n]}$ be a simplicial complex on the vertex set $[n] = \{1,\ldots,n\}$. We define that always $\emptyset \in \Delta$ is a face. To avoid trivial cases we assume throughout this chapter that $\Delta \neq 2^{[n]}$ and $n \geq 2$. The *(real) coordinate subspace arrangement* in $\mathbb{R}^n$ associated with Δ is

$$\mathcal{A}_\Delta = \{\operatorname{span}_{\mathbb{R}}\{e_{i_0},\ldots,e_{i_k}\} : \{i_0,\ldots,i_k\} \in \Delta\},$$

the *(complex) coordinate subspace arrangement* in $\mathbb{C}^n$ associated with Δ is

$$\mathcal{A}_\Delta^{\mathbb{C}} = \left\{\operatorname{span}_{\mathbb{C}}\{e_{i_0}^{\mathbb{C}},\ldots,e_{i_k}^{\mathbb{C}}\} : \{i_0,\ldots,i_k\} \in \Delta\right\}.$$

For every subspace arrangement we have the notion of the link and the complement, which in our case we denote by L_Δ and C_Δ, resp. $L_\Delta^{\mathbb{C}}$ and $C_\Delta^{\mathbb{C}}$.

$$L_\Delta = \mathbb{S}^{n-1} \cap \bigcup \mathcal{A}_\Delta \qquad C_\Delta = \mathbb{R}^n \setminus \bigcup \mathcal{A}_\Delta$$
$$L_\Delta^{\mathbb{C}} = \mathbb{S}^{2n-1} \cap \bigcup \mathcal{A}_\Delta^{\mathbb{C}} \qquad C_\Delta^{\mathbb{C}} = \mathbb{C}^n \setminus \bigcup \mathcal{A}_\Delta^{\mathbb{C}}$$

1.2.2 Models for the Real Case

We introduce combinatorial models Λ_Δ and Γ_Δ for L_Δ and C_Δ. Consider the n-dimensional cross polytope $Q^n = \operatorname{conv}\{\pm e_i : i = 1, \ldots, n\}$. Its proper faces form a simplicial complex, which we denote by ∂Q^n. Let Λ_Δ be the subcomplex of ∂Q^n of all simplices that are contained in $\bigcup \mathcal{A}_\Delta$.

$$\Lambda_\Delta = \left\{\{\varepsilon_0 e_{i_0}, \ldots, \varepsilon_k e_{i_k}\} : \{i_0, \ldots, i_k\} \in \Delta, (\varepsilon_0, \ldots, \varepsilon_k) \in \{\pm 1\}^{k+1}\right\}$$

Let Γ_Δ be the "mirror complex" of $\mathcal{A}_\Delta$ (cf. [BBC97]), i.e., the faces of the n-cube $C^n = [-1,1]^n$ disjoint to $\bigcup \mathcal{A}_\Delta$ considered as a polytopal subcomplex of the cube.

$$\Gamma_\Delta = \{c : c \text{ a proper face of } C^n,\ [n] \setminus \{\text{varying coord. of } c\} \notin \Delta\}$$

The underlying spaces $|\Lambda_\Delta|$ and $|\Gamma_\Delta|$ are homeomorphic, resp. homotopy equivalent, to the link L_Δ and the complement C_Δ, see e.g. [Mun84, p. 414].

1.2.3 From Complex to Real Arrangements

As far as the topology is concerned any complex coordinate arrangement can be modeled as a real subspace arrangement. Let $\Delta \subset 2^{[n]}$ be a simplicial complex on the vertex set $\{1, \ldots, n\}$. Let $\pi : [2n] \longrightarrow [n]$ the map defined by $2i-1, 2i \mapsto i$ for $i \in [n]$. Define the *"complexification" of* Δ by

$$\Delta^{\mathbb{C}} = \{\sigma \subset [2n] : \pi(\sigma) \in \Delta\}.$$

For an example of a "complexification" and the following Lemma see Figure 1.1.

Lemma 1.2.1.

- ▷ Under the standard identification $\mathbb{C}^n \cong \mathbb{R}^{2n}$ the spaces $\bigcup \mathcal{A}_\Delta^{\mathbb{C}}$ and $\bigcup \mathcal{A}_{\Delta^{\mathbb{C}}}$ correspond to each other.
- ▷ For $\sigma \in \Delta^{\mathbb{C}}$ the following homotopy equivalence holds

$$\operatorname{link}_{\Delta^{\mathbb{C}}} \sigma \simeq \begin{cases} * & \text{if } \pi^{-1}(\pi(\sigma)) \neq \sigma, \\ \operatorname{link}_\Delta \pi(\sigma) & \text{if } \pi^{-1}(\pi(\sigma)) = \sigma. \end{cases}$$

□

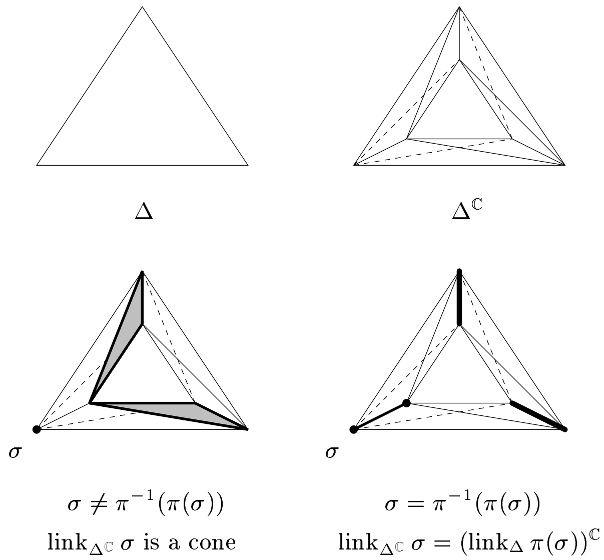


Figure 1.1: Example for the "complexification" of a complex Δ.

1.2.4 The Goresky–MacPherson Theorem

Let $\mathcal{A}$ be a (linear) subspace arrangement in $\mathbb{R}^n$ with link $L = \mathbb{S}^{n-1} \cap \bigcup \mathcal{A}$ and complement $C = \mathbb{R}^n \setminus \bigcup \mathcal{A}$. Denote by P the intersection poset of $\mathcal{A}$ ordered by reversed inclusion, and by $d{:}P \longrightarrow \mathbb{N}$ the dimension function. For $v \in P$ let $P_{<v}$ be the subposet of all elements in P that are smaller than v. For any finite poset Q denote by $\Delta(Q)$ the order complex of Q.

Theorem (Goresky–MacPherson [GM88, Part III]). *The homology of the link $L_{\mathcal{A}}$, and the cohomology of the complement $C_{\mathcal{A}}$, of a subspace arrangement $\mathcal{A}$ in $\mathbb{R}^n$ can be computed from the data (P, d) and n:*

$$\tilde{H}_i(L_{\mathcal{A}}; \mathbb{Z}) \cong \bigoplus_{v \in P} \tilde{H}_{i-d(v)}(\Delta(P_{<v}); \mathbb{Z}),$$
$$\tilde{H}^i(C_{\mathcal{A}}; \mathbb{Z}) \cong \bigoplus_{v \in P} \tilde{H}_{n-i-d(v)-2}(\Delta(P_{<v}); \mathbb{Z}).$$

This theorem, originally proven by means of stratified Morse theory in [GM88], was given an elementary proof by Ziegler and Živaljević in [ZŽ93].

1.2.5 The Goresky–MacPherson Theorem for coordinate subspace arrangements

In the situation of a real coordinate subspace arrangement $\mathcal{A}_\Delta$ the order complexes $\Delta(P_{<v})$ can be described more explicitly. The poset P is given by the face poset of the simplicial complex Δ ordered by inverse inclusion. The poset

$P_{<\sigma}$ then is isomorphic to the opposite face lattice of $\operatorname{link}_\Delta \sigma = \{\tau \in \Delta : \sigma \cup \tau \in \Delta, \sigma \cap \tau = \emptyset\}$. Thus we obtain the following formulation of the Goresky–MacPherson theorem.

Theorem. *Let $\Delta \subset 2^{[n]}$ be a simplicial complex with vertex set $\{1, \ldots, n\}$. Then*

$$\tilde{H}_i(L_\Delta; \mathbb{Z}) \cong \bigoplus_{\sigma \in \Delta} \tilde{H}_{i-|\sigma|}(\operatorname{link}_\Delta \sigma; \mathbb{Z}),$$
$$\tilde{H}^i(C_\Delta; \mathbb{Z}) \cong \bigoplus_{\sigma \in \Delta} \tilde{H}_{n-i-|\sigma|-2}(\operatorname{link}_\Delta \sigma; \mathbb{Z}).$$

Here $|\sigma|$ denotes the cardinality of σ, i.e., $|\sigma| = \dim \sigma + 1$.

In view of section 1.2.2 this yields the following result for the associated complex coordinate subspace arrangement.

Corollary 1.2.2. For simplicial complexes $\Delta \subset 2^{[n]}$ we have

$$\tilde{H}_i\left(L_\Delta^{\mathbb{C}}; \mathbb{Z}\right) \cong \bigoplus_{\sigma \in \Delta} \tilde{H}_{i-2|\sigma|}(\operatorname{link}_\Delta \sigma; \mathbb{Z})$$
$$\tilde{H}^i\left(C_\Delta^{\mathbb{C}}; \mathbb{Z}\right) \cong \bigoplus_{\sigma \in \Delta} \tilde{H}_{2n-i-2|\sigma|-2}(\operatorname{link}_\Delta \sigma; \mathbb{Z}),$$

and hence there is a dimension-shifting group isomorphism between the (co)homologies of the real and complex coordinate subspace arrangements. Every homology class

$$[c] \in \tilde{H}_{n-i-|\sigma|-2}(\operatorname{link}_\Delta \sigma; \mathbb{Z}) = \tilde{H}_{2n-(n+|\sigma|+i)-2|\sigma|-2}(\operatorname{link}_\Delta \sigma; \mathbb{Z})$$

corresponds to

$$[u] \in \tilde{H}^i(C_\Delta; \mathbb{Z})$$

and to

$$[u^{\mathbb{C}}] \in \tilde{H}^{n+|\sigma|+i}\left(C_\Delta^{\mathbb{C}}; \mathbb{Z}\right).$$

The correspondence $[u] \longmapsto [u^{\mathbb{C}}]$ sets up the isomorphism.

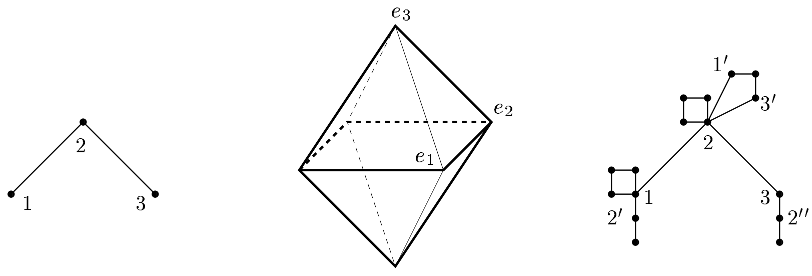
1.2.6 A Homology Model and a Map into the Link

We establish a simplicial version of the Ziegler–Živaljević [ZŽ93] proof for the Goresky–MacPherson theorem. Let $\Delta \subset 2^{[n]}$ be a simplicial complex. We construct a simplicial complex $\mathfrak{L}_\Delta$ together with a simplicial map $\Phi : \mathfrak{L}_\Delta \longrightarrow$

Λ_Δ to the link that induces an isomorphism in homology. Let $\mathfrak{L}_\Delta$ be the following one-point union of spaces.

$$\mathfrak{L}_\Delta = \left(\dot{\bigcup_{\sigma \in \Delta}} \partial Q^{|\sigma|} * \mathrm{link}_\Delta\, \sigma \right) /\sim \; = \left(\Delta \,\dot{\cup} \dot{\bigcup_{\sigma \in \Delta \setminus \{\emptyset\}}} \partial Q^{|\sigma|} * \mathrm{link}_\Delta\, \sigma \right) /\sim$$

The one-point union is given by the following identifications $\sim$. For each $\sigma = \{i_0 < \ldots < i_k\} \in \Delta$, $\sigma \neq \emptyset$, identify $e_1 \in \partial Q^{|\sigma|} * \mathrm{link}_\Delta\, \sigma$ with the vertex $i_0 \in \Delta = \partial Q^{|\emptyset|} * \mathrm{link}_\Delta\, \emptyset$. Compare Figure 1.2.



Δ $\qquad$ Λ_Δ $\qquad$ $\mathfrak{L}_\Delta$

Figure 1.2: An easy example for the model space $\mathfrak{L}_\Delta$.

We get the map Φ by defining it on the pieces $\partial Q^{|\sigma|} * \mathrm{link}_\Delta\, \sigma$. Let

$$\phi_\sigma : \partial Q^{|\sigma|} * \mathrm{link}_\Delta\, \sigma \longrightarrow \Lambda_\Delta$$

be defined by the simplicial homeomorphism

$$\partial Q^{|\sigma|} \longrightarrow \mathrm{span}_{\mathbb{R}}\{e_{i_0}, \ldots, e_{i_k}\} \cap \partial Q^n,$$

$\sigma = \{i_0 < \cdots < i_k\}$, such that $\phi_\sigma(e_{j+1}) = e_{i_j}$, in particular $\phi_\sigma(e_1) = e_{i_0}$. On $\mathrm{link}_\Delta\, \sigma$ the map ϕ_σ is defined by

$$\{j_0, \ldots, j_l\} \longmapsto \{e_{j_0}, \ldots, e_{j_l}\} \in \Lambda_\Delta$$

for $\{j_0, \ldots, j_l\} \in \mathrm{link}_\Delta\, \sigma$. By construction all these maps fit together and yield a simplicial map Φ.

Proposition 1.2.3. The map Φ induces an isomorphism in homology. (In fact, it is a homotopy equivalence.)

Sketch of proof. The proof works as in [ZŽ93] by induction on the cardinality of Δ. In the induction step one removes a maximal simplex of Δ and uses the Mayer-Vietoris sequence along with the induction hypotheses (resp. the Glueing Lemma, to obtain the homotopy equivalence). $\square$

1.2.7 Cubical Cohomology

The homotopy model Γ_Δ of the complement C_Δ is a subcomplex of the boundary of the cube. We compute its cohomology by using "cubical cohomology." We give a short overview of the most important notation and the formula for the cup product (see also [Mas91]).

Let Γ be a subcomplex of the n-cube C^n, and let $T \in \Gamma$ be a t-dimensional cube. We use two descriptions of T:

Denote the projection to the i-th coordinate by π_i. On the one hand, we can identify T with a vector in $\{+,-,*\}^n$, where the i-th coordinate is $+$, $-$ or $*$ iff $\pi_i(T) = \{+1\}$, $\{-1\}$, resp. $[-1,+1]$. On the other hand, there are three sets $T_+, T_-, T_* \subseteq \{1,\dots,n\}$ that uniquely define the cube,

$$T \overset{1-1}{\leftrightsquigarrow} (T_+, T_-, T_*),$$

where $|T_*| = t$ and the following holds for the coordinate projections.

$$\begin{aligned}
\pi_i(T) &= \{+1\} && \text{for } i \in T_+,\\
\pi_j(T) &= \{-1\} && \text{for } j \in T_-,\\
\pi_k(T) &= [-1,+1] && \text{for } k \in T_*.
\end{aligned}$$

Let $C_t(\Gamma)$ be the free abelian group generated by the t-cubes in Γ. In order to get a boundary map we begin by defining face operators. Let $T \in \Gamma$ be a t-dimensional cube $T \overset{1-1}{\leftrightsquigarrow} (T_+, T_-, T_*)$ with $T_* = \{k_1 < \cdots < k_t\}$. For $A = \{a_1, \dots, a_p\} \subseteq \{1,\dots,t\}$ and $\varepsilon = \pm 1$ define the $(t-p)$-cube

$$D_A^\varepsilon T = \begin{cases} (T_+ \cup \{k_{a_1},\dots,k_{a_p}\}, T_-, T_* \setminus \{k_{a_1},\dots,k_{a_p}\}) & \text{if } \varepsilon = +1,\\ (T_+, T_- \cup \{k_{a_1},\dots,k_{a_p}\}, T_* \setminus \{k_{a_1},\dots,k_{a_p}\}) & \text{if } \varepsilon = -1. \end{cases}$$

$D_A^\varepsilon T$ is the face of T obtained by fixing the varying coordinates $\{k_{a_1},\dots,k_{a_p}\}$ to ε. A boundary operator is now defined by

$$\begin{aligned}
\partial_t : C_t(\Gamma) &\longrightarrow C_{t-1}(\Gamma),\\
T &\longmapsto \sum_{a=1}^{t} (-1)^a \left(D_{\{a\}}^{+1} T - D_{\{a\}}^{-1} T \right).
\end{aligned}$$

The homology of the resulting *cubical chain complex* $(C_*(\Gamma), \partial_*)$ is canonically isomorphic to singular homology. The cup product formula in this situation is given on the chain level by the following. Let $u \in \mathrm{Hom}(C_p(\Gamma), \mathbb{Z})$ and $v \in \mathrm{Hom}(C_q(\Gamma), \mathbb{Z})$, then for a $(p+q)$-cube T we obtain

$$(u \cup v)(T) = \sum \rho_{H,K} \cdot u\left(D_H^{+1} T\right) v\left(D_K^{-1} T\right),$$

where the sum is taken over all q-subsets H of $\{1,\dots,p+q\}$, K is the complement of H, and $\rho_{H,K}$ is the sign of the permutation HK of $\{1,\dots,p+q\}$, i.e., the signature of the shuffle (H,K).

1.2.8 Lefschetz Duality for the Cross Polytope

As a crucial part of Alexander duality, we describe Lefschetz duality explicitly for simplicial homology of the cross polytope and cubical cohomology of the cube (cf. [Mun84]).

Theorem (Lefschetz Duality). *Let (X, A) be a compact, orientable, triangulated relative homology n-manifold. Then there is an isomorphism*

$$H_k(X, A) \cong H^{n-k}(|X| \setminus |A|).$$

Outline of the proof. Let X^- be the simplicial complex consisting of all simplices of the barycentric subdivision $\operatorname{sd} X$ that are disjoint from $|A|$. Then

- ▷ $|X^-|$ is a deformation retract of $|X| \setminus |A|$.
- ▷ $|X^-|$ equals the union of all blocks $D(\sigma)$ dual to simplices $\sigma \in X$ that are not in A.

Now there is a chain isomorphism

$$C^k(X, A) \xrightarrow{\cong} D_{n-k}(X^-),$$

where $D_*(X^-)$ denotes the dual chain complex of X^-. Dualization yields

$$C_k(X, A) \cong \operatorname{Hom}(C^k(X, A), \mathbb{Z}) \xleftarrow{\cong} \operatorname{Hom}(D_{n-k}(X^-), \mathbb{Z}).$$

The inverse map $C_k(X, A) \longrightarrow \operatorname{Hom}(D_{n-k}(X^-), \mathbb{Z})$ is given by $\sigma \mapsto D(\sigma)^*$, where σ is a k-simplex of X not in A. This induces the desired isomorphism. □

Lefschetz duality is dealing with the complex X^-, whose underlying space is the union of the dual blocks $D(\sigma)$, $\sigma \in X \setminus A$. In case X is the boundary of the cross polytope Q^n, the dual blocks $|D(\sigma)|$, $\sigma \in X$, correspond to the faces of the boundary of the n-dimensional cube C^n. See Figure 1.3.

Let now $A = \Lambda_\Delta$ be the subcomplex of $X = \partial Q^n$ given by the arrangement associated with a simplicial complex Δ (Section 1.2.2). Then there is a chain isomorphism from the dual block complex of $(\partial Q^n)^-$ to the cubical chain complex of Γ_Δ

$$D_j((\partial Q^n)^-) \longrightarrow C_j(\Gamma_\Delta),$$

which yields a chain isomorphism

$$\Psi : C_k(\partial Q^n, \Lambda_\Delta) \longrightarrow \operatorname{Hom}(D_{n-1-k}((\partial Q^n)^-), \mathbb{Z}) \longrightarrow \operatorname{Hom}(C_{n-1-k}(\Gamma_\Delta), \mathbb{Z})$$

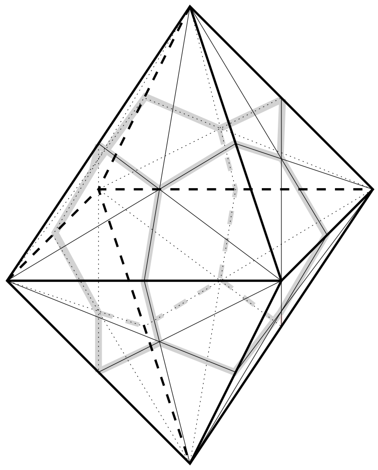


Figure 1.3: The 3-dimensional cross polytope with the 1-skeleton of the 3-dimensional cube in the barycentric subdivision.

where

$$\Psi(\sigma) = (-1)^{i_0+\cdots+i_k}(-1)^{|T_-(\sigma)|}(T_+(\sigma), T_-(\sigma), T_*(\sigma))^*,$$

for $\sigma = \langle \varepsilon_0 e_{i_0}, \ldots, \varepsilon_k e_{i_k} \rangle \in \partial Q^n \setminus \Lambda_\Delta$, $i_0 < \cdots < i_k$, with

$$\begin{aligned} T_+(\sigma) &= \{i_j \in [n] : \varepsilon_j = +1\}, \\ T_-(\sigma) &= \{i_j \in [n] : \varepsilon_j = -1\}, \\ T_*(\sigma) &= [n] \setminus (T_+(\sigma) \cup T_-(\sigma)). \end{aligned}$$

The signs in $\Psi(\sigma)$ result from the condition that Ψ must commute with the respective boundary maps.

1.3 Proofs of Results

In this section we prove Theorem 1.1.1. We begin by introducing joins of chains, and then exhibit explicit cohomology classes in $\tilde{H}^*(\Gamma_\Delta)$ with respect to the Goresky–MacPherson theorem. We derive an explicit formula for the cup product of two such classes. In most of the cases the product vanishes as stated in Theorem 1.1.1. Then we treat the case in which the product does not vanish. The considerations of the complex case follow then.

1.3.1 Joins of Chains

Definition 1.3.1. The *join* $c * c'$ *of two simplicial chains* $c = \sum_j \alpha_j \tau_j$ *and* $c' = \sum_k \alpha'_k \tau'_k$ in a simplicial complex $\Delta \subset 2^{[n]}$ is defined by

$$\sum_{\substack{j,k \\ \tau_j \cap \tau'_k = \emptyset}} \alpha_j \alpha'_k \, \tau_j * \tau'_k,$$

where the join of two disjoint oriented simplices is defined by

$$\langle v_0, \ldots, v_r \rangle * \langle w_0, \ldots, w_s \rangle = \langle v_0, \ldots, v_r, w_0, \ldots, w_s \rangle.$$

Lemma 1.3.2. Let $R = \{r_0, \ldots, r_s\}$ be a subset of the vertex set, $c = \sum_j \alpha_j \tau_j$ a cycle. For $R \subset \tau_j$ define the (oriented) simplex $\bar{\tau}_j$ by the equation $\tau_j = \bar{\tau}_j * \langle r_0, \ldots, r_s \rangle$. Then $\sum_{j : R \subset \tau_j} \alpha_j \bar{\tau}_j$ is a cycle.

Proof. We write c as

$$c = \sum_{j : R \not\subset \tau_j} \alpha_j \tau_j + \sum_{j : R \subset \tau_j} \alpha_j \bar{\tau}_j * \langle r_0, \ldots, r_s \rangle,$$

and obtain for the boundary

$$\partial \left(\sum_{j : R \not\subset \tau_j} \alpha_j \tau_j \right) + \partial \left(\sum_{j : R \subset \tau_j} \alpha_j \bar{\tau}_j \right) * \langle r_0, \ldots, r_s \rangle \\ \pm \sum_{j : R \subset \tau_j} \alpha_j \bar{\tau}_j * \partial(\langle r_0, \ldots, r_s \rangle) = 0.$$

The only simplices that contain R appear in the second summand, and hence this summand must be zero on its own. □

Lemma 1.3.3. Let i be a vertex and let $c = \sum_j \alpha_j \tau_j$ and $c' = \sum_k \alpha'_k \tau'_k$ be two cycles that share at most the vertex i. Then

$$\partial(\langle i \rangle * c * c') = c * c'.$$

Proof.

$$
\begin{aligned}
\partial(\langle i\rangle * c * c') &= \partial\left(\langle i\rangle * \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k\right)\\
&= \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k - \langle i\rangle * \partial\left(\sum_{j:i\notin\tau_j} \alpha_j\tau_j\right) * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k\\
&\quad \pm \langle i\rangle * \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \partial\left(\sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k\right)\\
&= \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k + \langle i\rangle * \partial\left(\sum_{j:i\in\tau_j} \alpha_j\tau_j\right) * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k\\
&\quad \pm \langle i\rangle * \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \partial\left(\sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k\right)\\
&= c * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k - \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \langle i\rangle * \partial\left(\sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k\right)\\
&= c * \sum_{k:i\notin\tau'_k} \alpha'_k\tau'_k + \sum_{j:i\notin\tau_j} \alpha_j\tau_j * \langle i\rangle * \partial\left(\sum_{k:i\in\tau'_k} \alpha'_k\tau'_k\right)\\
&= c * c',
\end{aligned}
$$

where possible empty sums are considered to be zero. $\square$

1.3.2 Explicit Cocycles

Using the Goresky–MacPherson theorem and the explicit description of Alexander duality we now derive explicit cohomology cocycles for the complement of a coordinate subspace arrangement. For that, we use the following sequence of homomorphisms.

$$
\begin{aligned}
\tilde{H}_r(\operatorname{link}_\Delta \sigma) &\xrightarrow[\text{suspension}]{\cong} \tilde{H}_{r+|\sigma|}\left(\partial Q^{|\sigma|} * \operatorname{link}_\Delta \sigma\right) \xrightarrow[(\phi_\sigma)_*]{\hookrightarrow} \tilde{H}_{r+|\sigma|}(\Lambda_\Delta)\\
&\xrightarrow[\text{pair sequence}]{} \tilde{H}_{r+|\sigma|+1}(\partial Q^n, \Lambda_\Delta) \xrightarrow[\text{Lefschetz duality}]{\cong} \tilde{H}^{n-r-|\sigma|-2}(\Gamma_\Delta) \qquad (1.1)
\end{aligned}
$$

Before describing the maps explicitly, we introduce some notation.

Notation 1.3.4. $\triangleright$ For each subset $\{j_1, \ldots, j_s\} \subset [n]$ we define

$$\operatorname{sign}(j_1 j_2 \cdots j_s) = \operatorname{sign}\pi,$$

where π is the permutation of $(1, \ldots, s)$ such that $j_{\pi(1)} < \cdots < j_{\pi(s)}$. For every family of subsets $A_1, \ldots, A_k \subset [n]$, where $A_i = \{j_1^i < \cdots < j_{m_i}^i\}$, we define

$$\operatorname{sign}(A_1 \cdots A_k) = \operatorname{sign}(j_1^1, \ldots, j_{m_1}^1, j_1^2, \ldots, j_{m_2}^2, \ldots, j_1^k, \ldots, j_{m_k}^k).$$

Furthermore, for every set $A = \{a_1, \ldots, a_k\} \subset [n]$ we abbreviate $(-1)^{a_1 + \cdots + a_n}$ by $(-1)^{\Sigma A}$.

- ▷ For each $\sigma \in \Delta$ let

$$s_\sigma = \sum_{\vec{\varepsilon} = (\varepsilon_0, \ldots, \varepsilon_k) \in \{\pm 1\}^{k+1}} \varepsilon_0 \cdots \varepsilon_k \cdot \langle \varepsilon_0 e_0, \ldots, \varepsilon_k e_k \rangle$$

be a generating simplicial cycle of $\tilde{H}_{|\sigma|-1}(\partial Q^{|\sigma|})$.

- ▷ For each $\sigma \in \Delta$ choose $i_\sigma \in [n] \setminus \sigma$ arbitrarily.

Now, let $\sigma \in \Delta$ and $[c] \in \tilde{H}_r(\operatorname{link}_\Delta \sigma)$, $c = \sum_j \alpha_j \tau_j$. Consider $\Phi: \mathfrak{L}_\Delta \longrightarrow \Lambda_\Delta$ as defined in Section 1.2.6 and the induced chain map $\Phi_\sharp : C_*(\mathfrak{L}_\Delta) \to C_*(\Lambda_\Delta)$. The first two steps in the sequence (1.1) of homomorphisms are given by

$$[c] \longmapsto [s_\sigma * c] \longmapsto [\Phi_\sharp(s_\sigma * c)].$$

Now we construct the pair sequence map. Consider the following "cone" over the chain $\Phi_\sharp(s_\sigma * c)$:

$$\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c).$$

Observation 1.3.5.

- ▷ $\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c) \in C_{r+|\sigma|+1}(\partial Q^n, \Lambda_\Delta)$ by the definition of Φ and i_σ,
- ▷ $\partial(\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c)) = \Phi_\sharp(s_\sigma * c)$ as a special case of Lemma 1.3.3, and
- ▷ for any $i'_\sigma \in [n] \setminus \sigma$, the cycles $\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c)$ and $\langle e_{i'_\sigma} \rangle * \Phi_\sharp(s_\sigma * c)$ in $C_{r+|\sigma|+1}(\partial Q^n, \Lambda_\Delta)$ are homologous.

Hence an element $[c] \in \tilde{H}_r(\operatorname{link}_\Delta \sigma)$ is mapped under (1.1) as follows.

$$\begin{aligned}[c] \longmapsto [s_\sigma * c] \longmapsto [\Phi_\sharp(s_\sigma * c)]& \\ \longmapsto [\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c)] &\longmapsto [\Psi(\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c))]\end{aligned}$$

The cocycle $\Psi(\langle e_{i_\sigma} \rangle * \Phi_\sharp(s_\sigma * c))$ is explicitly given by

$$\sum_{j : i_\sigma \notin \tau_j} \sum_{\vec{\varepsilon} \in \{\pm 1\}^{k+1}} \operatorname{sign}(i_\sigma \sigma \tau_j) \cdot (-1)^{i_\sigma + \sum \sigma + \sum \tau_j} \cdot \alpha_j \cdot \\ (T_+(j, \vec{\varepsilon}), T_-(j, \vec{\varepsilon}), T_*(j, \vec{\varepsilon}))^*,$$

where

$$\begin{aligned}
T_+(j,\vec{\varepsilon}) &= \tau_j \cup \{i_\sigma\} \cup \{i_l : \varepsilon_l = +1\},\\
T_-(j,\vec{\varepsilon}) &= \{i_l : \varepsilon_l = -1\},\\
T_*(j,\vec{\varepsilon}) &= [n] \setminus (T_+(j,\vec{\varepsilon}) \cup T_-(j,\vec{\varepsilon}))\\
&= [n] \setminus (\sigma \cup \tau_j \cup \{i_\sigma\}).
\end{aligned}$$

Here we made use of the equality $\varepsilon_0 \cdots \varepsilon_k \cdot (-1)^{|T_-(j,\vec{\varepsilon})|} = +1$. In the other representation the cubes $(T_+(j,\vec{\varepsilon}), T_-(j,\vec{\varepsilon}), T_*(j,\vec{\varepsilon}))$ look as in Figure 1.4 (up to a permutation of coordinates), where the $\pm$-signs correspond to the sign vector $\vec{\varepsilon}$.

$$(\underbrace{\pm\pm\pm\pm\pm\pm\pm\pm\pm}_{\sigma}|*\ *\ *\ *|\underbrace{\ +\ }_{\{i_\sigma\}}|*\ *\ *\ *\ *|\underbrace{++++}_{\tau_j}|*),$$

Figure 1.4: Schematic description of $(T_+(j,\vec{\varepsilon}), T_-(j,\vec{\varepsilon}), T_*(j,\vec{\varepsilon}))$.

Throughout the rest of the chapter we will use this correspondence between homology cycles of the links of Δ and cocycles of the complement of the arrangement.

1.3.3 The Cup Product

Now consider two cohomology classes $[u]$ and $[v]$ of Γ_Δ corresponding to two homology classes $[c] \in \tilde{H}_r(\mathrm{link}_\Delta\,\sigma)$ and $[c'] \in \tilde{H}_{r'}(\mathrm{link}_\Delta\,\sigma')$ for simplices $\sigma, \sigma' \in \Delta$, $c = \sum_j \alpha_j \tau_j$ and $c' = \sum_k \alpha'_k \tau'_k$. Let $p = n-r-|\sigma|-2$ and $q = n-r'-|\sigma'|-2$ and let $T \in \Gamma_\Delta$ be a $(p+q)$-cube. For the cup product of $[u]$ and $[v]$ evaluated at T we obtain

$$\sum_{H,K} \sum_{\substack{j: i_\sigma \notin \tau_j\\ k: i_{\sigma'} \notin \tau'_k}} \sum_{\vec{\varepsilon},\vec{\varepsilon}'} \rho_{H,K}\, \mathrm{sign}(i_\sigma \sigma \tau_j)\, \mathrm{sign}(i_{\sigma'} \sigma' \tau'_k) \cdot (-1)^{i_\sigma + i_{\sigma'} + \sum\sigma + \sum\tau_j + \sum\sigma' + \sum\tau'_k}.$$
$$\alpha_j \alpha'_k \cdot (T_+(j,\vec{\varepsilon}), T_-(j,\vec{\varepsilon}), T_*(j,\vec{\varepsilon}))^*(D_H^{+1}T)\cdot$$
$$(T'_+(k,\vec{\varepsilon}'), T'_-(k,\vec{\varepsilon}'), T'_*(k,\vec{\varepsilon}'))^*(D_K^{-1}T),$$

where the first summation is over all (q,p)-shuffles (H,K). Let us first consider only the last term

$$(T_+(j,\vec{\varepsilon}), T_-(j,\vec{\varepsilon}), T_*(j,\vec{\varepsilon}))^*(D_H^{+1}T)\cdot$$
$$(T'_+(k,\vec{\varepsilon}'), T'_-(k,\vec{\varepsilon}'), T'_*(k,\vec{\varepsilon}'))^*(D_K^{-1}T). \quad (*)$$

Observation 1.3.6. The term $(*)$ vanishes for all H, K and $\vec{\varepsilon}, \vec{\varepsilon}'$ unless

$$\sigma \cup \sigma' \cup \{i_\sigma\} \cup \{i_{\sigma'}\} \cup \tau_j \cup \tau'_k = [n].$$

Proof. The sets of varying coordinates in $D_H^{+1}T$ and $D_K^{-1}T$ are disjoint. This gives

$$\begin{aligned}\emptyset &= T_*(j, \vec{\varepsilon}) \cap T'_*(k, \vec{\varepsilon}') \\ &= ([n] \setminus (\sigma \cup \tau_j \cup \{i_\sigma\})) \cap ([n] \setminus (\sigma' \cup \tau'_k \cup \{i_{\sigma'}\})),\end{aligned}$$

which yields the result. $\square$

Now we turn to the general computation of the cup product.
Case I: $\sigma \neq \sigma'$ and $\sigma \cup \sigma' \neq [n]$.
In this case we will show that the cup product vanishes as demanded in the statement of the main Theorem. By anti-commutativity of the cup product we may assume $\sigma' \not\subset \sigma$. By Observation 1.3.5 we can assume $i_\sigma = i_{\sigma'} \notin [n]\setminus(\sigma\cup\sigma')$. Our situation is represented in Figure 1.5.

$$(*\ *|\overbrace{+\,+\,+\,+}^{\tau_j\cap\sigma'}|*\ *\ *\ *\ *|\overbrace{\pm\pm\pm\pm\pm\pm}^{\sigma}|*\ \overset{\downarrow}{*}\ *|\overbrace{+\,+\,+}^{\tau_j\setminus\sigma'}|\overbrace{\ +\ }^{i_\sigma})^*(D_H^{+1}T)\cdot$$

$$(\underbrace{\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm}_{\sigma'}|*\ *|\underbrace{+\,+\,+\,+\,+}_{\tau'_k}|*\ *|\underbrace{\ +\ }_{i_{\sigma'}})^*(D_K^{-1}T)$$

$$(*\ *|+\,+\,+\,+|*\ *\ *\ *\ *|\pm\,\pm\,\pm|*\ *|+|*\ *\ *|+|*\ *|\ +\ )$$

Figure 1.5: The term $(*)$ schematically and all cubes T for which $(*)$ has a chance not to vanish.

Observation 1.3.7. The term $(*)$ vanishes for all H, K, $\vec{\varepsilon}, \vec{\varepsilon}'$, and k unless the following holds

$$\sigma \cup \sigma' \cup \{i_\sigma\} \cup \tau_j = [n].$$

Proof. As the down arrow $\downarrow$ points out in Figure 1.5, if there is a coordinate only covered by τ'_k it will be a fixed -1-coordinate in $D_K^{-1}T$. $\square$

Hence all terms that have a chance to contribute to a non trivial product are as shown in Figure 1.6.

$$
\begin{array}{l}
(\ast\ \ast|\overbrace{+\ +\ +\ +}^{\tau_j\cap\sigma'}|\ast\ \ast\ \ast\ \ast\ \ast|\overbrace{\pm\ \pm\ \pm\ \pm\ \pm\ \pm}^{\sigma}|\overbrace{+\ +\ +\ +\ +\ +}^{\tau_j\setminus\sigma'}|\ \overbrace{+}^{i_\sigma}\ )^*(D_H^{+1}T)\cdot \\
\underbrace{(\pm\ \pm\ \pm\ \pm\ \pm\ \pm\ \pm\ \pm\ \pm\ \pm\ \pm\ \pm\ \pm\ \pm\ \pm}_{\sigma'}|\ast\ \ast|\underbrace{+\ +\ +\ +\ +}_{\tau'_k}|\ast\ \ast|\ \underbrace{+}_{i_{\sigma'}}\ )^*(D_K^{-1}T)
\end{array}
$$

$$
(\ast\ \ast|+\ +\ +\ +|\ast\ \ast\ \ast\ \ast\ \ast|\pm\ \pm\ \pm|\ast\ \ast|+\ +\ +\ +\ +|\ast\ \ast|\ +\ )
$$

Figure 1.6: The term $(\ast)$ schematically and all cubes T contributing non zero summands.

Gathering all contributing terms with the right sign and coefficient, we obtain that the cup product is represented by the following cocycle (up to a global sign)

$$
\Psi\left(\langle e_{i_\sigma}\rangle \ast \Phi_\sharp\left(s_{\sigma\cap\sigma'} \ast \sum_{\substack{j:i_\sigma\notin\tau_j\\ R\subset\tau_j}} \alpha_j\bar{\tau}_j \ast \sum_{k:i_\sigma\notin\tau'_k} \alpha'_k\tau'_k\right)\right),
$$

where $R=\{r_0,\dots,r_s\}=[n]\setminus(\sigma\cup\sigma'\cup\{i_\sigma\})$ and $\tau_j=\bar{\tau}_j\ast\langle r_0,\dots,r_s\rangle$. Tracing this element back through the sequence of homomorphisms in (1.1) and using Lemma 1.3.3 we arrive (up to a global sign) at

$$
\partial\left(\langle i_\sigma\rangle \ast \sum_{\substack{j:i_\sigma\notin\tau_j\\ R\subset\tau_j}} \alpha_j\bar{\tau}_j \ast \sum_{k:i_\sigma\notin\tau'_k} \alpha'_k\tau'_k\right) = \sum_{\substack{j\\ R\subset\tau_j}} \alpha_j\bar{\tau}_j \ast c'
$$

as a representing cycle in $\tilde{H}_*(\mathrm{link}_\Delta(\sigma\cap\sigma'))$, which we denote by $\bar{c}\ast c'$. This is a chain in $C_k(\mathrm{link}_\Delta(\sigma\cap\sigma'))$ for the following reason. Consider an arbitrary pair of simplices $\bar{\tau}_j$ and τ'_k. Since $\tau'_k\in\mathrm{link}_\Delta\sigma'$ we have $\sigma'\cup\tau'_k\in\Delta$ and since $\bar{\tau}_j\subset\sigma'$ we obtain $\bar{\tau}_j\cup\tau'_k\in\Delta$.
We claim that the cycle $\bar{c}\ast c'$ is a boundary in $C_k(\mathrm{link}_\Delta(\sigma\cap\sigma'))$. Since $\sigma'\not\subset\sigma$ there is a $p\in\sigma'\setminus\sigma$ and as before all simplices $\{p\}\cup\bar{\tau}_j\cup\tau'_k\in\Delta$. Hence $\langle p\rangle\ast\bar{c}\ast c'\in C_{k+1}(\mathrm{link}_\Delta(\sigma\cap\sigma'))$ with boundary $\bar{c}\ast c'$ as follows by Lemma 1.3.3.
Case II: $\sigma=\sigma'$ (of course $\sigma\neq[n]$). We will show that the cup product vanishes unless the complement C_Δ is not connected. In this case we get non-trivial self multiplication of elements in cohomological dimension 0. Again we can assume $i_\sigma=i_{\sigma'}$. Our situation is shown in Figure 1.7.

$$\Big(\overbrace{\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm}^{\sigma}\Big|\overset{\downarrow}{*}\;*\Big|\overbrace{+\;+\;+\;+}^{\tau_j}\Big|\overbrace{\;+\;}^{i_\sigma}\;\Big)^*(D_H^{+1}T)\cdot$$
$$\Big(\underbrace{\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm}_{\sigma'}\Big|\underbrace{+\;+\;+\;+}_{\tau'_k}\Big|*\;*\Big|\underbrace{\;+\;}_{i_{\sigma'}}\;\Big)^*(D_K^{-1}T)$$

$$\Big(\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\Big|*\;*\Big|+\;+\Big|*\;*\Big|\;+\;\Big)$$

Figure 1.7: The term $(*)$ schematically and all possible cubes T on which it does not vanish.

As in the last case $(\downarrow)$ the only interesting terms are the ones with $\sigma \cup \tau_j \cup \{i_\sigma\} = [n]$. If such a term exists c must be a multiple of a generating cycle of the sphere on the vertices $[n] \setminus \sigma$. But for c' not to be trivial the same holds for c', since the reduced homology of the sphere is non trivial only in the dimension of the sphere. Now $n - |\sigma| - r - 2 = n - |\sigma'| - r' - 2 = 0$. Therefore the corresponding cohomology classes are not zero only if C_Δ is not connected, which means that there are simplices of dimension $n - 2$ in Δ.

Case III: $\sigma \cup \sigma' = [n]$. Consider Figure 1.8.

$$\Big(\;*\;*\Big|\overbrace{+\;+\;+\;+}^{\tau_j}\Big|*\;*\Big|\overbrace{\;+\;}^{i_\sigma}\Big|*\;*\Big|\overbrace{\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\pm\;\pm}^{\sigma}\;\Big)^*(D_H^{+1}T)\cdot$$
$$\Big(\underbrace{\pm\pm\pm\pm\pm\pm\pm\pm\;\pm\;\pm\pm\pm\pm\pm}_{\sigma'}\Big|*\;*\Big|\underbrace{+\;+\;+\;+\;+}_{\tau'_k}\Big|*\;*\Big|\underbrace{\;+\;}_{i_{\sigma'}}\;\Big)^*(D_K^{-1}T)$$

$$\Big(*\;*\Big|+\;+\;+\;+\Big|*\;*\Big|\;+\;\Big|*\;*\Big|\pm\pm\pm\Big|*\;*\Big|+\;+\;+\;+\;+\Big|*\;*\Big|\;+\;\Big)$$

Figure 1.8: The term $(*)$ schematically and all cubes T for which $(*)$ does not vanish.

Gathering the cocubes corresponding to the non vanishing summands together with signs and coefficients gives the following representing cocycle for the cup product (up to a global sign, see Section 1.3.4)

$$\Psi\left(\langle e_{i_\sigma}\rangle * \langle e_{i_{\sigma'}}\rangle * \Phi_\sharp\left(s_{\sigma\cap\sigma'} * \sum_{j:i_\sigma\notin\tau_j} \alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k} \alpha'_k\tau'_k\right)\right).$$

Tracing this element back to a cycle in $C_{r+r'+2}(\mathrm{link}_\Delta(\sigma\cap\sigma'))$ leads up to a factor of $(-1)^{|\sigma\cap\sigma'|}$ to

$$
\begin{aligned}
&\partial\left(\langle i_\sigma\rangle * \langle i_{\sigma'}\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k\right) = \\
&= \langle i_{\sigma'}\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k - \langle i_\sigma\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k \\
&\quad - \langle i_{\sigma'}\rangle * \langle i_\sigma\rangle * \partial\left(\sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j\right) * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k \\
&\quad + \langle i_\sigma\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \langle i_{\sigma'}\rangle * \partial\left(\sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k\right) \\
&= \langle i_{\sigma'}\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k - \langle i_\sigma\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k \\
&\quad + \langle i_{\sigma'}\rangle * \langle i_\sigma\rangle * \partial\left(\sum_{j:i_\sigma\in\tau_j}\alpha_j\tau_j\right) * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k \\
&\quad - \langle i_\sigma\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \langle i_{\sigma'}\rangle * \partial\left(\sum_{k:i_{\sigma'}\in\tau'_k}\alpha'_k\tau'_k\right) \\
&= \langle i_{\sigma'}\rangle * c * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k - \langle i_\sigma\rangle * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * c' \\
&= \langle i_{\sigma'}\rangle * c * c' - \langle i_\sigma\rangle * c * c'.
\end{aligned}
$$

This finishes the proof of Theorem 1.1.1.

1.3.4 The Global Sign

We show how to compute the global sign. In the cup product formula we have the sign

$$\rho_{H,K}\,\mathrm{sign}(i_\sigma\sigma\tau_j)\,\mathrm{sign}(i_{\sigma'}\sigma'\tau'_k)\cdot(-1)^{i_\sigma+i_{\sigma'}+\sum\sigma+\sum\tau_j+\sum\sigma'+\sum\tau'_k}. \qquad (*)$$

For the image of $\langle i_{\sigma'}\rangle * c * c' - \langle i_\sigma\rangle * c * c'$ under the sequence of homomorphisms (1.1) we obtain

$$(-1)^{|\sigma\cap\sigma'|}\cdot\Psi\left(\langle e_{i_\sigma}\rangle * \langle e_{i_{\sigma'}}\rangle * \Phi_\sharp\left(s_{\sigma\cap\sigma'} * \sum_{j:i_\sigma\notin\tau_j}\alpha_j\tau_j * \sum_{k:i_{\sigma'}\notin\tau'_k}\alpha'_k\tau'_k\right)\right).$$

In this sum the sign of the cube in question is

$$(-1)^{|\sigma\cap\sigma'|}\cdot \operatorname{sign}(i_\sigma i_{\sigma'}(\sigma\cap\sigma')\tau_j\tau'_k)\cdot(-1)^{i_\sigma+i_{\sigma'}+\sum(\sigma\cap\sigma')+\sum\tau_j+\sum\tau'_k}. \qquad (**)$$

The global sign is given by the quotient of the two signs $(*)$ and $(**)$.

$$\begin{aligned}(-1)^{|\sigma\cap\sigma'|}&\cdot\rho_{H,K}\operatorname{sign}(i_\sigma\sigma\tau_j)\cdot\operatorname{sign}(i_{\sigma'}\sigma'\tau'_k)\cdot\operatorname{sign}(i_\sigma i_{\sigma'}(\sigma\cap\sigma')\tau_j\tau'_k)\cdot\\ &(-1)^{i_\sigma+i_{\sigma'}+\sum\sigma+\sum\tau_j+\sum\sigma'+\sum\tau'_k}\cdot(-1)^{i_\sigma+i_{\sigma'}+\sum(\sigma\cap\sigma')+\sum\tau_j+\sum\tau'_k}\\ &\quad=(-1)^{|\sigma\cap\sigma'|+\sum(\sigma\cup\sigma')}\cdot\rho_{H,K}\operatorname{sign}(i_\sigma\sigma\tau_j)\cdot\operatorname{sign}(i_{\sigma'}\sigma'\tau'_k)\cdot\\ &\qquad\qquad\operatorname{sign}(i_\sigma i_{\sigma'}(\sigma\cap\sigma')\tau_j\tau'_k),\end{aligned}$$

where

$$\begin{aligned}H&=[n]\setminus(\sigma'\cup\tau'_k\cup\{i_{\sigma'}\})\\ K&=[n]\setminus(\sigma\cup\tau_j\cup\{i_\sigma\}).\end{aligned}$$

We will derive a formula that is easier to handle and, in particular, shows the independence of j,k and $i_\sigma, i_{\sigma'}$.

Lemma 1.3.8. Let $\sigma,\sigma'\subset[n]$ such that $\sigma\cup\sigma'=[n]$, and $\iota=\{i\}\subset[n]\setminus\sigma$, $\iota'=\{i'\}\subset[n]\setminus\sigma'$, and $r,r'\geq 0$. Then for $\tau\subset[n]\setminus(\sigma\cup\iota)$ and $\tau'\subset[n]\setminus(\sigma'\cup\iota')$ of cardinality r, resp. r' we have

$$\begin{aligned}&\operatorname{sign}(([n]\setminus(\sigma'\cup\tau'\cup\iota'))([n]\setminus(\sigma\cup\tau\cup\iota)))\cdot\\ &\qquad\operatorname{sign}(\iota\sigma\tau)\operatorname{sign}(\iota'\sigma'\tau')\operatorname{sign}(\iota\iota'(\sigma\cap\sigma')\tau\tau')\\ &\qquad\qquad=(-1)^{rr'+r'(n-|\sigma|-1)+1}\operatorname{sign}(([n]\setminus\sigma')([n]\setminus\sigma)).\end{aligned}$$

Note that for simplicity we have used r,r' for the cardinalities of τ,τ' instead of the dimensions.

Proof. We proceed in two steps. First we show, what happens if we reduce (r,r') in the lexicographic order.
For $(r,r')=(0,0)$, we just have

$$\operatorname{sign}(([n]\setminus(\sigma'\cup\iota'))([n]\setminus(\sigma\cup\iota)))\cdot\operatorname{sign}(\iota\sigma)\operatorname{sign}(\iota'\sigma')\operatorname{sign}(\iota\iota'(\sigma\cap\sigma')). \quad (1.2)$$

Now assume $r=0$ and $r'>0$. Choose two r'-sets $\tau'_1,\tau'_2\subset[n]\setminus(\sigma'\cup\iota')$, and choose elements $v_1\in\tau'_1, v_2\in\tau'_2$. Let $\bar\tau'_1=\tau_1\setminus\{v_1\}$ and $\bar\tau'_2=\tau'_2\setminus\{v_2\}$. Then

$$\begin{aligned}&\operatorname{sign}(([n]\setminus(\sigma'\cup\tau'_{1/2}\cup\iota'))([n]\setminus(\sigma\cup\iota)))\\ &\quad=\operatorname{sign}([n]\setminus(\sigma'\cup\bar\tau'_{1/2}\cup\iota')[n]\setminus(\sigma\cup\iota))(-1)^{|\{a\in[n]\setminus(\sigma\cup\iota):a<v_{1/2}\}|},\\ &\operatorname{sign}(\iota'\sigma'\tau'_{1/2})=\operatorname{sign}(\iota'\sigma'\bar\tau'_{1/2})(-1)^{|\{a\in\iota'\cup\sigma':a>v_{1/2}\}|},\\ &\operatorname{sign}(\iota\iota'(\sigma\cap\sigma')\tau'_{1/2})=\operatorname{sign}(\iota\iota'(\sigma\cap\sigma')\bar\tau'_{1/2})(-1)^{|\{a\in\iota\cup\iota'\cup(\sigma\cap\sigma'):a>v_{1/2}\}|}.\end{aligned}$$

Consider the sum of the (-1)-exponents.

$$
\begin{aligned}
&|\{a \in [n] \setminus (\sigma \cup \iota) : a < v_{1/2}\}| + |\{a \in \iota' \cup \sigma' : a > v_{1/2}\}| + \\
&|\{a \in \iota \cup \iota' \cup (\sigma \cap \sigma') : a > v_{1/2}\}| \\
&\quad \equiv |[n] \setminus (\iota \cup \sigma)| - |\{a \in [n] \setminus (\sigma \cup \iota) : a > v_{1/2}\}| + \\
&\qquad |\{a \in \sigma' : a > v_{1/2}\}| + |\{a \in \iota \cup (\sigma \cap \sigma') : a > v_{1/2}\}| \\
&\quad \equiv |[n] \setminus (\iota \cup \sigma)| + |\{a \in \sigma' : a > v_{1/2}\}| + \\
&\qquad |\{a \in [n] \setminus (\sigma \cup \iota) : a > v_{1/2}\}| + |\{a \in \iota \cup (\sigma \cap \sigma') : a > v_{1/2}\}| \\
&\quad \equiv |[n] \setminus (\iota \cup \sigma)| + 2|\{a \in \sigma' : a > v_{1/2}\}| \\
&\quad \equiv |[n] \setminus (\iota \cup \sigma)| \pmod 2.
\end{aligned}
$$

Hence, for reducing r' by one we obtain a factor of $(-1)^{n-|\sigma|-1}$ and thus in total a factor $(-1)^{r'(n-|\sigma|-1)}$.

Assume $r > 0$. This case works analogously, reducing two choices of r-sets $\tau_{1/2}$. In each step one gets a factor $(-1)^{r'}$. Hence, after r steps, we obtain a factor $(-1)^{rr'}$.

Treating the expression (1.2) similarly yields

$$(-1) \cdot \operatorname{sign}(([n] \setminus \sigma')([n] \setminus \sigma)),$$

which gives the result. ☐

Thus, we derived the following global sign

$$
\begin{aligned}
&(-1)^{\frac{n(n+1)}{2}+|\sigma\cap\sigma'|+(r+1)(r'+1)+(r'+1)(n-|\sigma|-1)+1} \cdot \operatorname{sign}(([n] \setminus \sigma')([n] \setminus \sigma)) \\
&\qquad = (-1)^{\frac{n(n+1)}{2}+|\sigma\cap\sigma'|+(r'+1)(n+|\sigma|+r)+1} \cdot \operatorname{sign}(([n] \setminus \sigma')([n] \setminus \sigma)). \quad (1.3)
\end{aligned}
$$

1.3.5 The Complex Case

We will explicitly compute the multiplication in $\tilde{H}^*(C_\Delta^{\mathbb{C}};\mathbb{Z})$ using the results and notation of Section 1.2.3 and the previous Section. Let $[u], [v] \in \tilde{H}^*(C_\Delta^{\mathbb{C}};\mathbb{Z})$ correspond to

$$[c] \in \tilde{H}_r(\operatorname{link}_\Delta \sigma) \cong \tilde{H}_r(\operatorname{link}_{\Delta^{\mathbb{C}}} \pi^{-1}(\sigma))$$

and

$$[c'] \in \tilde{H}_{r'}(\operatorname{link}_\Delta \sigma') \cong \tilde{H}_{r'}(\operatorname{link}_{\Delta^{\mathbb{C}}} \pi^{-1}(\sigma'))$$

for simplices $\sigma, \sigma' \in \Delta$.

Case I: If $\sigma \cup \sigma' \neq [n]$ then $\pi^{-1}(\sigma) \cup \pi^{-1}(\sigma') \neq [2n]$ and hence the cup product of $[u]$ and $[v]$ is zero.

Case II: If $\sigma = \sigma' \neq [n]$ the cup product vanishes since the complement of a

complex coordinate subspace arrangement is connected.
Case III: Now let $\sigma \cup \sigma' = [n]$. Consider the isomorphism

$$\begin{aligned} \tilde{H}_r(\operatorname{link}_\Delta \sigma) &\longrightarrow \tilde{H}_r(\operatorname{link}_{\Delta^{\mathbb{C}}} \pi^{-1}(\sigma)) \\ [c] &\longmapsto [c^{\mathbb{C}}] \end{aligned}$$

induced by the vertex map $i \mapsto 2i-1$. It corresponds to the isomorphism induced by the homotopy equivalence. Using this isomorphism for the cup product computation we are in the well known situation as shown in Figure 1.9.

$$\begin{array}{l} (* \; * | \overbrace{+ + + +}^{\tau_j^{\mathbb{C}}} | * \; * | \overbrace{\;+\;}^{i_\sigma^{\mathbb{C}}} | * \; * \; * | \overbrace{\pm \pm \pm \pm \pm \pm \pm \pm \pm \pm \pm \pm \;\; \pm}^{\pi^{-1}(\sigma)} \;\;)^*(D_H^{+1}T) \cdot \\ (\underbrace{\pm \pm \pm \pm \pm \pm \pm \pm \;\; \pm \;\; \pm \pm \pm \pm \pm}_{\pi^{-1}(\sigma')} | * \; * | \underbrace{+ + + +}_{\tau_k'^{\mathbb{C}}} | * \; * \; * | \underbrace{\;+\;}_{i_{\sigma'}^{\mathbb{C}}} \;\;)^*(D_K^{-1}T) \end{array}$$

$$(* \; * | + + + + | * \; * | \; + \; | * \; * \; * | \pm \pm | * \; * | + + + + | * \; * \; * | \; + \;\;)$$

Figure 1.9: A typical summand of the cup product evaluated at T schematically and the cubes T for which it does not vanish.

Collecting all summands yields the cocycle

$$\Psi \left(\langle e_{i_\sigma} \rangle * \langle e_{i_{\sigma'}} \rangle * \Phi_\sharp \left(s_{\pi^{-1}(\sigma) \cap \pi^{-1}(\sigma')} * \sum_{j : i_\sigma \notin \tau_j^{\mathbb{C}}} \alpha_j \tau_j^{\mathbb{C}} * \sum_{k : i_{\sigma'} \notin \tau_k'^{\mathbb{C}}} \alpha_k' \tau_k'^{\mathbb{C}} \right) \right)$$

for vertices $i_\sigma \in [2n] \setminus \pi^{-1}(\sigma)$ and $i_{\sigma'} \in [2n] \setminus \pi^{-1}(\sigma')$. As above this leads (up to the global sign) to

$$\begin{aligned} [\langle i_{\sigma'} \rangle * c * c' - \langle i_\sigma \rangle * c * c'] &\in \tilde{H}_{r+r'+2}(\operatorname{link}_\Delta \sigma \cap \sigma') \\ &\cong \tilde{H}_{r+r'+2}(\operatorname{link}_{\Delta^{\mathbb{C}}} \pi^{-1}(\sigma \cap \sigma')) \\ &= \tilde{H}_{r+r'+2}(\operatorname{link}_{\Delta^{\mathbb{C}}} \pi^{-1}(\sigma) \cap \pi^{-1}(\sigma')). \end{aligned}$$

1.3.6 The Global Sign in the Complex Case

First of all, from the computation in the real case, we obtain the sign

$$(-1)^{n(2n+1)+|\pi^{-1}(\sigma) \cap \pi^{-1}(\sigma')|+(r+1)(r'+1)+(r'+1)(2n-|\pi^{-1}(\sigma)|-1)+1} \cdot \\ \operatorname{sign}(([2n] \setminus \pi^{-1}(\sigma'))([2n] \setminus \pi^{-1}(\sigma))).$$

Now in $\pi^{-1}(\sigma)$, $\pi^{-1}(\sigma')$ resp., all elements appear in pairs. This simplifies the sign to

$$(-1)^{n+r(r'+1)+1}. \tag{1.4}$$

1.4 Example of a Simplicial Complex yielding different Ring Structures

Let $[u], [v], [w]$ be cohomology classes of the complement of a real coordinate subspace arrangement corresponding to homology classes of links of Δ, such that $[u] \cup [v] = [w]$. Then our results imply that for the corresponding cohomology classes of the complement of the associated complex arrangement we have (see Corollary 1.2.2)

$$\left[u^{\mathbb{C}}\right] \cup \left[v^{\mathbb{C}}\right] = \pm \left[w^{\mathbb{C}}\right].$$

Hence it arises the question if we can choose signs in the correspondence $[u] \mapsto [u^{\mathbb{C}}]$ consistently such that it becomes a (dimension-shifting) ring isomorphism. An example of different ring structures containing hyperplanes was given in [GPW98]: the existence of hyperplanes lead to additional multiplication in the real case. Our example shows that this is not the only case where non-isomorphic rings occur.

Remark 1.4.1. There is a (dimension shifting) ring isomorphism of $\tilde{H}^*(C_\Delta; \mathbb{Z}_2)$ and $\tilde{H}^*(C_\Delta^{\mathbb{C}}; \mathbb{Z}_2)$.

1.4.1 The Example: Different Sign Patterns

We construct a simplicial complex $\Delta \subset 2^{[8]}$ on eight vertices given by four facets $\sigma_1, \sigma_2, \sigma_1', \sigma_2'$, and investigate the multiplication of cohomology classes stemming from the links of these facets in the case of the associated real and complex arrangement. For the real and complex case the resulting sign pattern implies that there is no ring isomorphism between $\tilde{H}^*(C_\Delta)$ and $\tilde{H}^*(C_\Delta^{\mathbb{C}})$. The facets are given by the following scheme which also helps for computing the signs appearing in the multiplication. A black box in position (ρ, j) indicates that $j \in \rho$.

	1	2	3	4	5	6	7	8
σ_1	■	■	■	■	■	■		
σ_2	■		■	■	■			■
σ_1'	■	■			■	■	■	■
σ_2'		■	■	■		■	■	■

Figure 1.10: The facets of Δ.

The sign patterns arising in the real and in the complex case according to (1.3) and (1.4) are given by the following table.

		Sign (1.3)	Sign (1.4)
σ_1	σ_1'	-1	-1
σ_1	σ_2'	-1	-1
σ_2	σ_1'	$+1$	-1
σ_2	σ_2'	-1	-1

Clearly, there is no consistent way of assigning signs in the correspondence $[u] \mapsto [u^{\mathbb{C}}]$.

1.5 Example of non trivial multiplication of Torsion Elements

We construct a simplicial complex $\Delta \subset 2^{[10]}$. Let $\sigma := \{1,2,3,4,5,6\}$ and $P \subset 2^{\{1,2,3,4,5,6\}}$ be a six-vertex triangulation of the projective plane. Let $\sigma' = \{7,8,9,10\}$, and let S be a simplicial 1-sphere on four vertices as a subcomplex of $2^{\{7,8,9,10\}}$. Now define $\Delta = P * 2^{\sigma'} \cup 2^{\sigma} * S$. Then the homotopy type of Δ is $\Sigma(P * 2^{\sigma'} \cap 2^{\sigma} * S) = \Sigma(P * S)$. Hence Δ has the homotopy type of a threefold suspended projective plane. Now $\mathrm{link}_\Delta(\sigma * \emptyset) = \emptyset * S$ and $\mathrm{link}_\Delta(\emptyset * \sigma') = P * \emptyset$. Let $[c] \in \tilde{H}_1(\mathrm{link}_\Delta(\sigma * \emptyset)) \cong \mathbb{Z}$ and $[c'] \in \tilde{H}_1(\mathrm{link}_\Delta(\emptyset * \sigma')) \cong \mathbb{Z}_2$ be generating homology classes. They correspond to elements $[u] \in \tilde{H}^{10-1-6-2}(\Gamma_\Delta)$ and $[v] \in \tilde{H}^{10-1-4-2}(\Gamma_\Delta)$. Their cup product corresponds to a generating class

$$[\langle i_{\sigma'} \rangle * c * c' - \langle i_\sigma \rangle * c * c'] \in \tilde{H}_{10-4-0-2}(\mathrm{link}_\Delta \emptyset) \cong \mathbb{Z}_2$$

for $i_\sigma \in \{7,8,9,10\}$ and $i_{\sigma'} \in \{1,2,3,4,5,6\}$.
Note that this example works for the real as well as for the complex case.

1.6 Remarks

It is easy to see that if $\Delta \subset 2^{[n]}$ is a simplicial complex such that

- ▷ $\dim \Delta \leq n-3$, i.e., the associated real arrangement does not contain hyperplanes, and
- ▷ Δ is Cohen-Macaulay over $\mathbb{Z}$,

then the ring structure of $\tilde{H}^*(C_\Delta; \mathbb{Z})$ is trivial. Using the specific description of the multiplication it would be nice to derive a better characterization of simplicial complexes yielding trivial multiplication. Confer also [HRW99].

CHAPTER 2

THE COHOMOLOGY RINGS OF COMPLEMENTS OF GENERAL SUBSPACE ARRANGEMENTS

This chapter is joint work with Carsten Schultz [LS99].

2.1 Introduction

The integral cohomology's ring structure of complements of real linear subspace arrangements is the concern of this chapter. In order to put our results in the right context we recall some previously achieved results.

- ▷ Using *rational models* De Concini and Procesi derived that the multiplicative structure of the rational cohomology in the case of complex arrangements is determined by combinatorial data: intersection lattice and dimension function [CP95]. Their techniques were applied by Yuzvinsky to give an explicit description for the *rational cohomology ring* for complex arrangements [Yuz98].

- ▷ Generalizing the Orlik-Solomon result on complex hyperplane arrangements [OS80] Feichtner and Ziegler obtained a presentation for the *integral cohomology ring* of the complement of a complex arrangement with *geometric intersection lattice* [FZ00] by extending combinatorial stratification methods from Björner and Ziegler [BZ92]. Independently, Yuzvinsky obtained this result as an application of his work on the rational cohomology rings of complex arrangements mentioned above [Yuz98], [Yuz99].

- ▷ Ziegler gave a presentation for the integral cohomology ring of a *real* 2-*arrangement* [Zie93]. Applying this result he showed that intersection lattice and dimension function as combinatorial data do not suffice to determine the ring.

In this chapter we

- ▷ describe the *integral cohomology ring structure* for general real arrangements up to an error term,
- ▷ determine the *integral cohomology ring structure* for (≥ 2)-arrangements, a class *generalizing complex arrangements and real* 2-*arrangements,*
- ▷ give a presentation for the integral cohomology ring of (≥ 2)-arrangements with *geometric intersection lattice.*

Having Ziegler's result in mind we *extend the combinatorial data by orientation information* in the general case, i.e., all spaces in the arrangement are considered to have a specific orientation. Since all complex spaces inhabit a canonical orientation the orientation information becomes unnecessary in the case of complex arrangements. In this special case our result on the integral cohomology ring structure was conjectured by Yuzvinsky [Yuz98].

Apart from our new results this chapter *unifies* the results and *simplifies* the methods compared to the previously known: we employ elementary methods from combinatorics and topology only.

Our results are based on the description of the homology of the link by Goresky and MacPherson [GM88]. In fact, we describe purely combinatorially, i.e., using the *intersection lattice, the dimension function, and the orientation information,* a ring structure on this homology. The main result is that this combinatorially defined ring coincides with the cohomology ring of the arrangement in case of a (≥ 2)-arrangement. Its proof relies on the homotopy model of the link of an arrangement given by Ziegler and Živaljević [ZŽ93]. An important step is the insight that for (≥ 2)-arrangements all "standard" homotopy equivalences of the model to the link are homotopic. A crucial distinction in the description of the ring structure is given by a certain codimension condition: all computations of cohomology rings of subspace arrangements that have been done in the past lead to the impression that cohomology classes multiply trivially as long as they do not satisfy such a condition (cf., e.g., [OS80], [BZ92], [FZ00]).

After this work was finished we learned about the recent work of Deligne, Goresky and MacPherson [DGM99], where similar questions are considered. By a sheaf theoretic approach using derived categories they obtain – among others – comparable results.

2.1.1 Statement of results

The main theorem is concerned with the description of the integral cohomology ring of the complement of (≥ 2)-arrangements. The description is based on a formula by Goresky and MacPherson. If $\mathcal{A}$ is a real subspace arrangement with intersection poset P ordered by inverse inclusion then the homology of the link $L_{\mathcal{A}}$ and the cohomology of the complement $M_{\mathcal{A}}$ via Alexander duality is given by

$$\bigoplus_{u \in P} \tilde{H}_{r-\dim(u)}(\Delta(P_{<u})) \xrightarrow{\cong} \tilde{H}_r(L_{\mathcal{A}}) \xrightarrow{\cong} \tilde{H}^{n-r-2}(M_{\mathcal{A}}).$$

The combinatorial data give rise to the definition of a ring structure on the group on the left. This combinatorially defined ring will be compared with the cohomology ring of the arrangement. In the following theorem $*$ can be thought of as the topological join and is made precise in Section 2.5.

Theorem. *Let $\mathcal{A}$ be a (≥ 2)-arrangement of oriented linear subspaces in $\mathbb{R}^n$. The ring structure of the integral cohomology of the complement of $\mathcal{A}$ is given by the combinatorial data via*

$$\begin{aligned} \tilde{H}_r(\Delta(P_{<u})) \otimes \tilde{H}_s(\Delta(P_{<v})) &\longrightarrow \tilde{H}_{r+s+2}(\Delta(P_{<u\cap v})) \\ a \otimes b &\longmapsto \begin{cases} \varepsilon(\langle v \rangle * a * b - \langle u \rangle * a * b), & \text{if } u + v = \mathbb{R}^n \\ 0, & \text{else.} \end{cases} \end{aligned}$$

The sign ε is given by the orientations of u, v, and the various dimensions and is made explicit in Remark 2.5.2.

In the case of complex arrangements this result proves a conjecture by Yuzvinsky. Its geometrical proof is based on the homotopy model of the link defined by Ziegler and Živaljević [ZŽ93]. The following is the crucial step for the proof of the theorem.

Proposition. Let $\mathcal{A}$ be a (≥ 2)-arrangement then any two model maps from the homotopy model to the link are homotopic.

For general real linear subspace arrangements the last two statements are false, but nevertheless some statement can be made about the integral cohomology ring.

Theorem. *Let $\mathcal{A}$ be a general linear subspace arrangement. Then there is a filtration of the homology of the link, such that the associated graded abelian group G carries a ring structure induced by the ring structure of the cohomology of the complement, and G is ring isomorphic to the combinatorially defined ring associated with the arrangement.*

Again the driving force behind this theorem is a proposition about the homological difference of two model maps.

2.1.2 Organization of the chapter

In Section 2.2 we first introduce notation. In particular, we define our notion of combinatorial data which is somewhat non-standard because of the fact that all spaces are considered to be oriented. We proceed by recapitulating the construction of the Ziegler and Živaljević model of the link and the Goresky and MacPherson isomorphism. Section 2.3 gathers two easy facts about the homotopy models that may not have appeared before which we will need. The subsequent Section 2.4 deals with the homology of lattices, and various products laying the foundation for the definition of the ring associated with the combinatorial data of an arrangement, which happens in Section 2.5. The reader might want to skip Section 2.6 when reading this chapter for the first time, since here standard topological methods about duality and intersection products are gathered and combined coherently to have a notion of a linking product, which is what the reader will think it is. The heart of this chapter lies in Section 2.7 and 2.8. In these sections the linking product of classes is described depending on the validity of the codimension condition. The theorem for the class of (≥ 2)-arrangements is now achieved in Section 2.9 without much trouble. The section concludes with a presentation for the cohomology ring of a (≥ 2)-arrangement with geometric intersection lattice. Section 2.10 is an example section that demonstrates the importance of the restriction to (≥ 2)-arrangements. Finally, in the last Section 2.11 we will see what still can be said in the general case.

2.2 Preliminaries about arrangements

2.2.1 Notation

Let $\mathcal{A}$ be an *oriented* (linear) subspace arrangement in $\mathbb{R}^n$, i.e., a finite family of *oriented* linear subspaces of $\mathbb{R}^n$. The objective of this chapter is to relate topological and combinatorial data.

Topological data

We denote the associated *link* $\mathbb{S}^{n-1} \cap \bigcup \mathcal{A}$ by $L_{\mathcal{A}}$ and the associated *complement* $\mathbb{R}^n \setminus \bigcup \mathcal{A}$ by $M_{\mathcal{A}}$. The data we are interested in is the *homology* $\tilde{H}_*(L_{\mathcal{A}})$ *of the link* and the *cohomology* $\tilde{H}^*(M_{\mathcal{A}})$ *of the complement.* They are related via Alexander duality. In particular, we are interested in the ring structure of the cohomology given by the $\cup$-product.

Combinatorial data

As mentioned in the introduction our combinatorial data is slightly extended. It is given by:

▷ The set of all intersections of elements in $\mathcal{A}$ partially ordered by inverse

inclusion: the *intersection poset of* $\mathcal{A}$, denoted by P. It has a maximal element $\top := \bigcap \mathcal{A}$. The associated lattice is then $\hat{P} = P \cup \{\bot\}$, where $\bot = \mathbb{R}^n$.
▷ $\hat{P}$ is furnished with a *dimension function* $\dim\colon \hat{P} \to \mathbb{N}$ which assigns to each space its real dimension.
We assume that the following supplementary data is also given, i.e., we consider oriented arrangements.
▷ There is a *sign function*

$$\epsilon : \{(u, v) \in P \times P : u + v = \mathbb{R}^n\} \longrightarrow \{\pm 1\}$$

defined as follows. Let $u, v \in P$ such that $u + v = \mathbb{R}^n$. Consider an orientation frame $a_1, \ldots, a_r$ of the oriented space $u \cap v \in P$. This can be completed with some vectors $b_1, \ldots, b_s$ to be an orientation frame in u, resp. with some other vectors $c_1, \ldots, c_t$ to be an orientation frame in v. The family $a_1, \ldots, a_r, c_1, \ldots, c_t, b_1, \ldots, b_s$ of these vectors yields an orientation of $\mathbb{R}^n$. If this orientation coincides with the standard orientation then set $\epsilon_{u,v}$ equal to $+1$, otherwise define it to be -1.

Specific arrangements

Complex arrangements, i.e., finite families of linear subspaces of $\mathbb{C}^n$, will be considered as special real subspace arrangements in $\mathbb{R}^{2n}$. In this case all elements of the intersection poset are oriented canonically and the sign function ϵ only takes the value $+1$. Therefore, for complex arrangements the orientation data is superfluous. A much larger class of subspace arrangements is given by the family of (≥ 2)-arrangements.

Definition 2.2.1. An arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is called a (≥ 2)-*arrangement* if $\dim(u) - \dim(v) \geq 2$ for all $u, v \in \hat{P}$ with v a proper subspace of u.

2.2.2 The Ziegler-Živaljević homotopy model of the link

Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{R}^n$ with intersection poset P. Since the proof of our theorem relies on it, we will – à la Ziegler and Živaljević [ZŽ93] – describe a homotopy model of its link $L_{\mathcal{A}}$ and a map inducing a homotopy equivalence of the model to the link. This is one of the driving theorems describing topological data — the homotopy type of the link — by combinatorial data.

Order complexes

For any partially ordered set Q we denote by $\Delta(Q)$ *the order complex of* Q. It is the (abstract) simplicial complex on the vertex set Q, whose simplices are given by chains in Q, i.e., sets of pairwise comparable elements.

The model space

The homotopy model space $\mathcal{L}_{\mathcal{A}}$ of the link $L_{\mathcal{A}}$ is given by the one–point union

$$\mathcal{L}_{\mathcal{A}} = \coprod_{u \in P} \mathbb{S}^{\dim(u)-1} * \Delta(P_{<u}) \;/\; \sim \;,$$

where $\sim$ identifies the vertex $u \in \Delta(P_{<\top}) \subset \mathbb{S}^{\dim(\top)-1} * \Delta(P_{<\top})$ with the standard basis vector $e_1 \in \mathbb{S}^{\dim(u)-1} \subset \mathbb{S}^{\dim(u)-1} * \Delta(P_{<u})$. In the case $\top = 0$ we use the conventions $\mathbb{S}^{\dim(\top)-1} = \mathbb{S}^{-1} = \emptyset$ and $\emptyset * X = X * \emptyset = X$ for any space X.

Ingredients for model maps

For $u \in P$ let $\mathbb{S}_u$ be the $(\dim(u) - 1)$-dimensional sphere

$$\mathbb{S}_u = \mathbb{S}^{n-1} \cap u \subset \mathbb{R}^n .$$

It inherits an orientation from u.
▷ For $u \in P$, let $\iota_u : \mathbb{S}^{\dim(u)-1} \to \mathbb{S}_u$ be an orientation preserving homotopy equivalence such that $\iota_u(e_1) = x_u$ for $u \neq \top$. If $\top = 0$ the map ι_0 is just the empty map. Here $\mathbb{S}^{\dim(u)-1}$ is assumed to have the standard orientation coming from the standard orientation of $\mathbb{R}^{\dim(u)}$.
▷ In each of these spheres choose points: for $u \in P$ let $x_u \in \mathbb{S}_u \setminus \bigcup_{v>u} \mathbb{S}_v$. We refer to these points $x = (x_u)_{u<\top}$ as *generic points.*

The model maps

We construct a *model map* $\Phi^x : \mathcal{L}_{\mathcal{A}} \to L_{\mathcal{A}}$. It is given by maps

$$\iota_u * \phi_u : \mathbb{S}^{\dim(u)-1} * \Delta(P_{<u}) \to L_{\mathcal{A}}$$

on each of the pieces of the model space $\mathcal{L}_{\mathcal{A}}$. It remains to define the maps ϕ_u. Let ϕ_u on the vertices of $\Delta(P_{<u})$ be given by $\phi_u(v) = x_v$. For a chain $v_0 < v_1 < \cdots < v_k < u$ the set $\{\phi_u(v_0), \ldots, \phi_u(v_k)\} \subset \mathbb{S}_{v_0}$ defines a geodesic simplex. Let ϕ_u on the simplex $\{v_0, \ldots, v_k\}$ be the geodesic embedding onto $\{\phi_u(v_0), \ldots, \phi_u(v_k)\}$. This defines the map ϕ_u. Since for $u \in \Delta(P_{<\top}) \subset \mathbb{S}^{\dim(\top)-1} * \Delta(P_{<\top})$ and $e_1 \in \mathbb{S}^{\dim(u)-1} \subset \mathbb{S}^{\dim(u)-1} * \Delta(P_{<u})$ we have

$$\iota_\top * \phi_\top(u) = \phi_\top(u) = x_u = \iota_u(e_1) = \iota_u * \phi_u(e_1),$$

all the maps $\iota_u * \phi_u$ fit together and yield a model map Φ^x.

Proposition 2.2.2 ([ZŽ93] and [Sch97]). Any model map $\Phi^x : \mathcal{L}_{\mathcal{A}} \to L_{\mathcal{A}}$, as described above, is a homotopy equivalence.

By the construction of the model space we immediately have:

Corollary 2.2.3. The following diagram defines an isomorphism Ψ^x which we will call a Goresky-MacPherson isomorphism.

$$\begin{tikzcd} \bigoplus_{u\in P} \tilde{H}_{*-\dim(u)}(\Delta(P_{<u})) \arrow[r, "\Sigma", "\cong"'] \arrow[dr, dashed, "\Psi^x"'] & \bigoplus_{u\in P} \tilde{H}_{*}(\mathbb{S}^{\dim(u)-1} * \Delta(P_{<u})) \arrow[d, "\Phi^x_*", "\cong"'] \\ & \tilde{H}_*(L_{\mathcal{A}}) \end{tikzcd}$$

Moreover, Φ^x_* induces a double grading on the homology of the link $L_{\mathcal{A}}$ by assigning the degree (r, u) to a homology class in the image of $\tilde{H}_r(\mathbb{S}^{\dim(u)-1} * \Delta(P_{<u}))$ under Φ^x_*.

Note that the construction of a model depends on the choice of generic points. Indeed the homotopy class of a model map depends on this choice and even the induced Goresky-MacPherson isomorphism does. Fortunately, this dependence vanishes if we are dealing with (≥ 2)-arrangements. More details will be given in Section 2.3, 2.9 and 2.10.

2.2.3 The codimension condition

As already mentioned the following definition leads to a crucial distinction when describing the ring structure of the cohomology of an arrangement.

Definition 2.2.4. Let $\mathcal{A}$ be a subspace arrangement, and let $u, v \in P$. We will say that u and v satisfy the *codimension condition* if they are in general position, i.e., $\operatorname{codim} u + \operatorname{codim} v = \operatorname{codim} u \cap v$, which is equivalent to $u + v = \mathbb{R}^n$.

2.3 More about arrangements

2.3.1 Homotopic model maps

The following lemma shows that certain perturbations of a choice of generic points lead to homotopic model maps.

Lemma 2.3.1. Let x and x' be choices of generic points for an arrangement $\mathcal{A}$. Assume that $x_u = x'_u$ for all $u \in P$ with $u \neq v$. Furthermore, assume that there is a *path* $p : [0,1] \longrightarrow \mathbb{S}_v \setminus \bigcup_{u>v} \mathbb{S}_u$ *of generic points* connecting x_v and x'_v. Then the model maps Φ^x and $\Phi^{x'}$ are homotopic.

Proof. The homotopy is given by Φ^{x^t}, where x^t is the family of generic points given by $x^t_u := x_u$ for $u \neq v$ and $x^t_v := p(t)$. Since the path p is a path of generic points Φ^{x^t} is a well defined model map for all $t \in [0,1]$. Since for a fixed choice of generic points the definition of a model map is unique Φ^{x^t} will depend continuously on t. $\square$

2.3.2 Products with euclidean space

The following easy fact, together with Proposition 2.6.23 will allow us to restrict ourselves to classes in $\Psi^x\left(\tilde{H}_*(\Delta(P_{<u})\right)$ and $\Psi^x\left(\tilde{H}_*(\Delta(P_{<v}))\right)$ with $u \cap v = 0$ when calculating products.

Proposition 2.3.2. Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$, identify $\mathbb{R}^{m+n}$ with $\mathbb{R}^m \times \mathbb{R}^n$ and accordingly $\mathbb{S}^{m+n-1}$ with $\mathbb{S}^{m-1} * \mathbb{S}^{n-1}$. For $u \in \mathcal{A}$ set $u' := \mathbb{R}^m \times u$ and let $\mathcal{A}' := \{u' \colon u \in \mathcal{A}\}$ be the arrangement in $\mathbb{R}^{m+n}$ corresponding to $\mathcal{A}$ and having link $L_{\mathcal{A}'} = \mathbb{S}^{m-1} * L_{\mathcal{A}}$. Orient $\mathbb{S}_{u'}$ by $[\mathbb{S}^{m-1}] * [\mathbb{S}_u]$.

Assume that generic points $\{x_u\}$ for $\mathcal{A}$ and $\{x'_{u'}\}$ for $\mathcal{A}'$ are chosen in such a way that $x'_{u'} \in \mathbb{S}^{m-1} * \{x_u\}$ for all $u \in \mathcal{A}$. Let $h \colon \mathbb{S}^{m-1} * \mathcal{L}_{\mathcal{A}} \to \mathcal{L}_{\mathcal{A}'}$ be the homotopy equivalence defined by identifying $\Delta(P_{<u})$ with $\Delta(P'_{<u'})$ and $\mathbb{S}^{m-1} * \mathbb{S}^{\dim(u)-1}$ with $\mathbb{S}^{\dim(u')-1}$. Then $\Phi' \circ h \simeq \mathrm{id}_{\mathbb{S}^{m-1}} * \Phi$.

Proof. Move the points. □

2.4 A product for order complexes

We will now describe a product on the homology of certain order complexes that will be used to describe the ring structure on the homology of the link of an arrangement.

Throughout this section, let P and Q be lattices. We will denote all top elements by $\top$ and all bottom elements by $\perp$. As usual the least upper bound of elements a and b will be denoted by $a \vee b$ and their greatest lower bound by $a \wedge b$.

Note that $\Delta(P \times Q)$ is just $\Delta(P) \times \Delta(Q)$ endowed with the usual simplicial structure that a product of two simplicial complexes is given. Therefore, there is the well known map

$$C_*(P) \otimes C_*(Q) \xrightarrow{\times} C_*(P \times Q),$$

and on the right side of the arrow no harmful confusion is possible. It is given by

$$\langle u_0, \ldots, u_r \rangle \otimes \langle v_0, \ldots, v_s \rangle \mapsto \sum_{\substack{0=i_0 \leq \cdots \leq i_{r+s}=r \\ 0=j_0 \leq \cdots \leq j_{r+s}=s \\ (i_{k-1},j_{k-1}) \neq (i_k,j_k)\ \forall k}} \sigma_{i,j} \langle (u_{i_0}, v_{j_0}), \ldots, (u_{i_{r+s}}, v_{j_{r+s}}) \rangle$$

for chains $u_0 < u_1 < \cdots < u_r$ and $v_0 < v_1 < \cdots < v_s$, where the $\sigma_{i,j}$ are signs determined by $\sigma_{i,j} = 1$ if $k = 0$ or $l = 0$ and by $\mathfrak{d}(a \times b) = \mathfrak{d}a \times b + (-1)^r a \times \mathfrak{d}b$.

Although $\Delta(P)$ itself is contractible, this product induces useful products in homology, the most import one for our purposes being

$$\begin{aligned} &H_{k+2}(P, P_{<\top} \cup P_{>\perp}) \otimes H_{l+2}(Q, Q_{<\top} \cup Q_{>\perp}) \\ &\qquad \to H_{k+l+2+2}(P \times Q, (P \times Q)_{<\top} \cup (P \times Q)_{>\perp}). \end{aligned}$$

The reader be warned that we have written $P_{<\top} \cup P_{>\bot}$ for $\Delta(P_{<\top}) \cup \Delta(P_{>\bot})$, although really we should not have done so. The above dimensions have been chosen because of $\tilde{H}_k\left(P_{\substack{<\top\\>\bot}}\right) \cong H_{k+2}(P, P_{<\top} \cup P_{>\bot})$, since $\Delta(P)$ is a cone over a cone over $\Delta(P_{<\top} \cup P_{>\bot})$. Let us fix an isomorphism

$$C_{\top\bot}\colon \tilde{H}_k\left(P_{\substack{<\top\\>\bot}}\right) \to H_{k+2}(P, P_{<\top} \cup P_{>\bot})$$

which is given on the chain level by

$$\langle u_0, \ldots, u_k\rangle \mapsto (-1)^{k+2}\langle \bot, u_0, \ldots, u_k, \top\rangle.$$

The sign is chosen to make it possible to think of this map as $a \mapsto \langle \top, \bot\rangle * a$.

$C_{\top\bot}$ can be decomposed into two isomorphisms

$$\tilde{H}_k\left(P_{\substack{<\top\\>\bot}}\right) \xrightarrow{C_\bot} H_{k+1}\left(P_{<\top}, P_{\substack{<\top\\>\bot}}\right) \xrightarrow{C_\top} H_{k+2}(P, P_{<\top} \cup P_{>\bot})$$

or

$$\tilde{H}_k\left(P_{\substack{<\top\\>\bot}}\right) \xrightarrow{C_\top} H_{k+1}\left(P_{>\bot}, P_{\substack{<\top\\>\bot}}\right) \xrightarrow{-C_\bot} H_{k+2}(P, P_{<\top} \cup P_{>\bot})$$

where

$$\begin{aligned} C_\bot(\langle u_0, \ldots, u_k\rangle) &:= \langle \bot, u_0, \ldots, u_k\rangle, \\ C_\top(\langle u_0, \ldots, u_k\rangle) &:= (-1)^{k+1}\langle u_0, \ldots, u_k, \top\rangle. \end{aligned}$$

Note that in both of the compositions the first arrow is just the inverse of $\mathfrak{d}$.

This leads to the following

Definition 2.4.1. For lattices P, Q we define

$$\lambda\colon \tilde{H}_k\left(P_{\substack{<\top\\>\bot}}\right) \otimes \tilde{H}_l\left(Q_{\substack{<\top\\>\bot}}\right) \to \tilde{H}_{k+l+2}\left((P \times Q)_{\substack{<\top\\>\bot}}\right)$$

by

$$a \otimes b \mapsto C_{\top\bot}^{-1}(C_{\top\bot}(a) \times C_{\top\bot}(b)).$$

2.5 A ring defined by the combinatorial data

In this section we define the ring that under favourable circumstances will be isomorphic to the homology of the link of an arrangement endowed with the linking product.

Remember, that if P is the intersection poset of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$, it always contains a top element, and that by $\hat{P}$ we denote $P \cup \{\mathbb{R}^n\}$, which is a lattice with $\perp = \mathbb{R}^n$.

Now if $u, v \in P_{<0}$ span the whole of $\mathbb{R}^n$, the map $\vee\colon \hat{P}_{\leq u} \times \hat{P}_{\leq v} \to \hat{P}_{\leq u \vee v}$ is a monomorphism (and even an isomorphism, if all elements of $\mathcal{A}$ contain u or v) and induces a map

$$\vee_*\colon \tilde{H}_*\left((\hat{P}_{\leq u} \times \hat{P}_{\leq v})_{\substack{<(u,v)\\ >\perp}}\right) \to \tilde{H}_*(P_{<u\vee v}).$$

This allows for the following

Definition 2.5.1. Let $\hat{P}$ be the intersection lattice of an arrangement and let ϵ, dim be the additional combinatorial data.

We will define a ring structure on $R = \bigoplus_{u \in P} H_*(\Delta(P_{<u}))$ via the product

$$\circ\colon \tilde{H}_r(\Delta(P_{<u})) \otimes \tilde{H}_s(\Delta(P_{<v})) \to \tilde{H}_{r+s+2}(\Delta(P_{<u\vee v})),$$

which we call the *combinatorial linking product*, by

$$a \circ b := \begin{cases} \epsilon_{u,v}(-1)^{n+r(n-\dim v)} \vee_* (\lambda(a \otimes b)), & \\ & \text{if } \dim(u \vee v) = \dim u + \dim v - n, \\ 0, & \text{otherwise.} \end{cases}$$

Here, of course, $n = \dim \perp$.

Note that in the case of an arrangement with all dimensions even, for $\dim(u \vee v) = \dim u + \dim v - n$ this reduces to $a \circ b = \epsilon_{u,v} \vee_* (\lambda(a \otimes b))$ and for a complex arrangement even to $a \circ b = \vee_*(\lambda(a \otimes b))$. This is exactly the product defined in [Yuz98].

Remark 2.5.2. Defining $a * b$ for suitable chains and homology classes by

$$C_\perp(a * b) = \vee_*(C_\perp a \times C_\perp b)$$

we have

$$-C_\perp C_{u\vee v}(\vee_* \lambda(a \otimes b)) = \vee_*(C_{u\perp} a \times C_{v\perp} b) = C_\perp(C_u a * C_v b),$$

which yields

$$\begin{aligned} \vee_* \lambda(a \times b) = \mathfrak{d}(C_{u\vee v} \vee_* \lambda(a \otimes b)) = -\mathfrak{d}(C_u a * C_v b) = \\ = -\mathfrak{d}(u * a * v * b) = (-1)^{|a|}\mathfrak{d}(u * v * a * b) \end{aligned}$$

and therefore

$$a \circ b = \begin{cases} \epsilon_{u,v}(-1)^{n+r(n-\dim v+1)}(-\langle u\rangle + \langle v\rangle) * a * b, & \\ & \text{if } \dim(u \vee v) = \dim u + \dim v - n, \\ 0, & \text{otherwise.} \end{cases}$$

This relates our definition to the formulae obtained in Chapter 1 (respectively in [Lon00]) and to the Introduction 2.1.

2.6 Topological preliminaries

In this section we gather some topological facts, for the largest part from homology theory, that we will use. Since we are interested in links of arrangements and the unit sphere in a product of euclidean spaces is a join of the unit spheres in the spaces involved, we have to deal with joins if we consider products of arrangements. We treat joins first.

Our approach to the cohomolgy of the complement of an arrangement is via the homology of the link. Since these are isomorphic via Alexander duality and we are interested in the cup product in cohomology, we have to consider the corresponding product on homology, which we call the linking product. The largest part of this section is devoted to the study of the linking product, which we prepare by taking a short look at the related intersection product.

The material in this section is fairly standard, but sign and orientation conventions of different authors do not always agree, so we have tried to be explicit about them. Those here should coincide with those in [Dol72].

All spaces, pairs, etc. will be expected to fulfill niceness properties not stated explicitly. Finite simplicial complexes will always suffice.

2.6.1 Joins of spaces and of homology classes

Definition 2.6.1. The *join* $X * Y$ of two non-empty spaces X and Y is the quotient of the product $X \times I \times Y$ by the equivalence relation generated by $(x, 0, y) \sim (x, 0, y')$ and $(x, 1, y) \sim (x', 1, y)$ for all $x, x' \in X$, $y, y' \in Y$. We also define $X * \emptyset := X$, $\emptyset * Y := Y$. For pairs of spaces we set $(X, A) * (Y, B) := (X * Y, A * Y \cup X * B)$.

Note 2.6.2. Care is needed, because in general $(X, \emptyset) * (Y, \emptyset) \neq (X * Y, \emptyset)$.

Let p denote the quotient map $(X, A) \times (I, \mathfrak{d}I) \times (Y, B) \to (X, A) * (Y, B)$.

Proposition 2.6.3. The induced map

$$p_*\colon H_*((X, A) \times (I, \mathfrak{d}I) \times (Y, B)) \to H_*((X, A) * (Y, B))$$

is an isomorphism. For $p \in X$, $q \in Y$ the map

$$\tilde{H}_*(X * Y) \to H_*((X, \{p\}) * (Y, \{q\}))$$

is an isomorphism. □

Definition 2.6.4. For $a \in H_r(X, A)$, $b \in H_r(Y, B)$, we define $a * b \in H_{r+s+1}((X, A) * (Y, B))$ by $a * b := p_*(a \times [I] \times b)$, where $[I]$ is the generator of $H_1(I, \mathfrak{d}I)$ defined by $\mathfrak{d}[I] = -\langle 0 \rangle + \langle 1 \rangle$.

Remark 2.6.5. This definition is consistent with defining the join of two simplices by juxtaposition of vertices.

Definition 2.6.6. We define a join product for reduced homology by commutativity of

$$\begin{CD} \tilde{H}_*(X) \otimes \tilde{H}_*(Y) @>{\cong}>> H_*(X, \{p\}) \otimes H_*(Y, \{q\}) \\ @V{*}VV @VV{*}V \\ \tilde{H}_*(X * Y) @>{\cong}>> H_*\big((X, \{p\}) * (Y, \{q\})\big), \end{CD}$$

if $X \neq \emptyset$, $Y \neq \emptyset$, and by requiring that the join with the positive generator of $\tilde{H}_{-1}(\emptyset)$ be the identity otherwise. Similarly we define a join product for the reduced homology of a space and the homology of a pair.

Proposition 2.6.7. The above product does not depend on the choice of points p and q and is therefore well-defined. □

Some of the following propositions will be formulated for pairs only, even if we also need the corresponding statements about reduced homology.

Proposition 2.6.8. Let $t\colon X * Y \to Y * X$ be the homeomorphism $[x, \lambda, y] \mapsto [y, 1-\lambda, x]$ and $A \subset X$, $B \subset Y$, $a \in H_*(X, A)$, $b \in H_*(Y, A)$. Then $t_*(a * b) = (-1)^{(|b|+1)(|a|+1)} b * a$.

Proof. The corresponding statement for the homology cross product is [Dol72, Chap. VII, Eq. 2.8]. □

Proposition 2.6.9. Consider the homeomorphisms

$$\begin{aligned} h_1\colon \mathfrak{d}(D^k \times D^l) &\to S^{k+l-1} \\ x &\mapsto \frac{x}{\|x\|} \end{aligned}$$

and

$$\begin{aligned} h_2\colon S^{k-1} * S^{l-1} &\to S^{k+l-1} \\ [x, \lambda, y] &\mapsto \frac{((1-\lambda)x, \lambda y)}{\|((1-\lambda)x, \lambda y)\|}. \end{aligned}$$

The diagram

$$\begin{CD} H_k(D^k, S^{k-1}) \otimes H_l(D^l, S^{l-1}) @>{\mathfrak{d}\otimes\mathfrak{d}}>> H_{k-1}(S^{k-1}) \otimes H_{l-1}(S^{l-1}) \\ @V{\times}VV @VV{*}V \\ H_{k+l}(D^k \times D^l, \mathfrak{d}(D^k \times D^l)) @. H_{k+l-1}(S^{k-1} * S^{l-1}) \\ @V{\mathfrak{d}}VV @VV{h_{2*}}V \\ H_{k+l-1}(\mathfrak{d}(D^k \times D^l)) @>{h_{1*}}>> H_{k+l-1}(S^{k+l-1}) \end{CD}$$

commutes, and therefore, given orientations on D^k and D^l, the two orientations of S^{k+l-1} indicated by the diagram coincide. □

Proposition 2.6.10. Let $P = \{p\}$ be a space containing exactly one point. For a pair (X, A) define $C(X, A) := (P, \emptyset) * (X, A) = (P * X, X \cup P * A)$. Then for compact pairs (X_i, A_i), $i \in \{1, 2\}$ the diagram

$$\begin{tikzcd}
H_*(X_1, A_1) \otimes H_*(X_2, A_2) \arrow[r, "(\langle p\rangle *)\otimes(\langle p\rangle *)", "\cong"'] \arrow[dd, "*"] & H_*(C(X_1, A_1)) \otimes H_*(C(X_2, A_2)) \arrow[d, "\times"] \\
 & H_*(C(X_1, A_1) \times C(X_2, A_2)) \arrow[d, "f_*", "\cong"'] \\
H_*((X_1, A_1) * (X_2, A_2)) \arrow[r, "\langle p\rangle *", "\cong"'] & H_*(C((X_1, A_1) * (X_2, A_2))),
\end{tikzcd}$$

where

$$f\colon C(X_1, A_1) \times C(X_2, A_2) \to C((X_1, A_1) * (X_2, A_2))$$
$$\big([p, \lambda_1, x_1], [p, \lambda_2, x_2]\big) \mapsto \begin{cases} \big[0, \lambda_2, [x_1, \frac{\lambda_1}{2\lambda_2}, x_2]\big], & \lambda_2 \geq \lambda_1, \\ \big[0, \lambda_1, [x_2, 1 - \frac{\lambda_2}{2\lambda_1}, x_1]\big], & \lambda_1 \geq \lambda_2, \end{cases}$$

commutes. □

2.6.2 The intersection product

Let M^m be a compact oriented manifold, $[M] \in H_m(M)$ the orientation class.

Definition 2.6.11. For $A \subset X \subset M$ we denote by D the Poincaré-Alexander-Lefschetz duality isomorphism

$$D\colon H_r(X, A) \to H^{m-r}(M - A, M - X)$$

given by

$$D(a) \cap [M] = a.$$

Definition 2.6.12. For $A \subset X \subset M$ and $B \subset Y \subset M$ we define the intersection product

$$\bullet\colon H_r(X, A) \otimes H_s(Y, B) \to H_{r+s-m}(X \cap Y, (A \cup B) \cap (X \cap Y))$$

by requiring that

$$D(a \bullet b) = D(a) \cup D(b).$$

Proposition 2.6.13. We have

(i) $a \bullet b = (-1)^{(m-|a|)(m-|b|)} b \bullet a$,

(ii) $[M] \bullet a = a \bullet [M] = a$.

Proof. Compare [Dol72, VII.8.7,VII.8.9,VIII.13.8,VIII.13.10]. □

Proposition 2.6.14. Let N^n be another compact oriented manifold, and let $M \times N$ be oriented by

$$[M \times N] = \epsilon([M] \times [N]),$$

$\epsilon \in \{-1, +1\}$. For a, c and b, d homology classes of appropriate subspaces of M respectively N (or pairs) we have

$$(a \times b) \bullet (c \times d) = \epsilon \cdot (-1)^{(n-|b|)(m-|c|)}(a \bullet c) \times (b \bullet d).$$

Proof. Compare [Dol72, VII.8.16,VIII.13.13] □

Using Proposition 2.6.3 we have as an immediate consequence the following

Proposition 2.6.15. Let $A_i \subset X_i \subset \mathbb{S}^k$, $B_i \subset Y_i \subset \mathbb{S}^l$, $A_i, B_i \neq \emptyset$, $a_i \in H_*(X_i, A_i)$, $b_i \in H_*(Y_i, B_i)$ for $i \in \{0, 1\}$. Then

$$(a_0 * b_0) \bullet (a_1 * b_1) = \epsilon \cdot (-1)^{(l-|b_0|)(k-|a_1|)}(a_0 \bullet a_1) * (b_0 \bullet b_1),$$

if $[\mathbb{S}^k * \mathbb{S}^l] = \epsilon([\mathbb{S}^k] * [\mathbb{S}^l])$ □

2.6.3 The linking product

The exact definition of Alexander duality in dimensions 0 and $n-1$ varies. The following form seems to be best suited to our application.

Definition 2.6.16. Let $X \subset \mathbb{S}^{n-1}$, $q \in \mathbb{S}^{n-1} \backslash X$. By AD_q we denote the Alexander duality isomorphism

$$AD_q\colon \tilde{H}_r(X) \to H^{n-2-r}(\mathbb{S}^{n-1} \backslash X, \{q\})$$

given by

$$AD_q(\mathfrak{d}a) = D(a)$$

ie the inverse of the composite

$$H^{n-2-r}(\mathbb{S}^{n-1} \backslash X, \{q\}) \xrightarrow[\cong]{\cap[\mathbb{S}^{n-1}]} H_{r+1}(\mathbb{S}^{n-1} \backslash \{q\}, X) \xrightarrow[\cong]{\mathfrak{d}} \tilde{H}_r(X).$$

We will often simply write AD instead of AD_q.

Definition 2.6.17. For $X, Y \subset \mathbb{S}^{n-1}$, $q \in \mathbb{S}^{n-1} \backslash (X \cup Y)$ we define the *linking product*

$$\circ\colon \tilde{H}_r(X) \otimes \tilde{H}_s(Y) \to \tilde{H}_{r+s+2-n}(X \cup Y)$$

by

$$AD_q(a \circ b) = AD_q(a) \cup AD_q(b).$$

Note 2.6.18. If either r or s equals $n-2$, the linking product depends on the point q. However, it only depends on the connected component of $\mathbb{S}^{n-1}\backslash(X \cup Y)$ in which q lies.

Remark 2.6.19. For the special case $r+s=n-2$, X, Y disjoint connected oriented manifolds of dimensions r and s, this yields just the familiar linking number.

Proposition 2.6.20. We have $a \circ b = \mathfrak{d}((\mathfrak{d}^{-1}a) \bullet (\mathfrak{d}^{-1}b))$.

Proof. $AD(a \circ b) = AD(a) \cup AD(b) = D(\mathfrak{d}^{-1}a) \cup D(\mathfrak{d}^{-1}b) = D((\mathfrak{d}^{-1}a) \bullet (\mathfrak{d}^{-1}b)) = AD(\mathfrak{d}((\mathfrak{d}^{-1}a) \bullet (\mathfrak{d}^{-1}b)))$. □

The next proposition states that unlinked subspaces have trivial linking product. We will use it in Section 2.8.

Proposition 2.6.21. Let $X_i \subset \mathbb{S}^{n-1}$ for $i=0,1$. If there exist $B_i \supset X_i$ such that the maps $\tilde{H}_{r_i}(X_i) \to \tilde{H}_{r_i}(B_i)$ induced by inclusion are trivial and $B_0 \cap B_1 = \emptyset$ and $q \notin B_0 \cup B_1$, then the linking product $\tilde{H}_{r_0}(X_0) \otimes \tilde{H}_{r_1}(X_1) \to \tilde{H}_{r_0+r_1+2-n}(X_0 \cup X_1)$ is trivial.

Proof. The linking product in question is the composition of the second column of the following commutative diagram.

$$\begin{tikzcd}
H_{r_0+1}(B_0, X_0) \otimes H_{r_1+1}(B_1, X_1) \arrow[r, two heads, "\mathfrak{d}\otimes\mathfrak{d}"] \arrow[d, "\bullet"'] \arrow[dr] & \tilde{H}_{r_0}(X_0) \otimes \tilde{H}_{r_1}(X_1) \\
H_{r_0+r_1+3-n}(\emptyset) \arrow[dr] & H_{r_0+1}(\mathbb{S}^{n-1}\backslash\{q\}, X_0) \otimes H_{r_1+1}(\mathbb{S}^{n-1}\backslash\{q\}, X_1) \arrow[u, "\mathfrak{d}\otimes\mathfrak{d}", "\cong"'] \arrow[d, "\bullet"'] \\
 & H_{r_0+r_1+3-n}(\mathbb{S}^{n-1}\backslash\{q\}, X_0 \cup X_1) \arrow[d, "\mathfrak{d}"'] \\
 & \tilde{H}_{r_0+r_1+2-n}(X_0 \cup X_1)
\end{tikzcd}$$

The surjectivity of the map in the first row follows from the long exact sequences of the pairs (B_i, X_i), and of course $H_*(\emptyset) \cong 0$. □

Corollary 2.6.22. If $X_i \subset \mathbb{S}^{n-1}$, $i=1,2$, lie on different sides of an $(n-2)$-sphere embedded in $\mathbb{S}^{n-1}$, then the linking product $\tilde{H}_{r_1}(X_1) \otimes \tilde{H}_{r_2}(X_2) \to \tilde{H}_{r_1+r_2+2-n}(X_1 \cup X_2)$ is trivial for $r_i < n-2$.

Proof. Choose the B_i to be the components of the complement of the $(n-2)$-Sphere in $\mathbb{S}^{n-1}\backslash\{q\}$. Each of them is homotopy equivalent to either a point or $\mathbb{S}^{n-2}$ and therefore $\tilde{H}_{r_i}(B_i) \cong 0$. □

2.6.4 Linking products in joins of spheres

The following constructions are dual to the exterior cohomology product.

Proposition 2.6.23. Let $X \subset \mathbb{S}^{n-1}$. Consider $\mathbb{S}^{m-1} * X \subset \mathbb{S}^{m-1} * \mathbb{S}^{n-1}$, orient $\mathbb{S}^{m+n-1} = \mathbb{S}^{m-1} * \mathbb{S}^{n-1}$ by $[\mathbb{S}^{m-1}] * [\mathbb{S}^{n-1}]$, and use $q \in \mathbb{S}^{n-1} \backslash X$ to define Alexander Duality in $\mathbb{S}^{n-1}$ as well as $\mathbb{S}^{m+n-1}$. Then the map

$$\begin{aligned} j\colon \tilde{H}_*(X) &\to \tilde{H}_*(\mathbb{S}^{m-1} * X) \\ a &\mapsto (-1)^m [\mathbb{S}^{m-1}] * a \end{aligned}$$

is an isomorphism shifting degree by m and respecting linking products.

Proof. That j is an isomorphism shifting degrees, if not clear anyway, follows from Proposition 2.6.3 and the Künneth theorem. Additionally we have

$$\begin{aligned} j(a \circ b) &= (-1)^m [\mathbb{S}^{m-1}] * \mathfrak{d}(\mathfrak{d}^{-1} a \bullet \mathfrak{d}^{-1} b) \\ &= \mathfrak{d}([\mathbb{S}^{m-1}] * (\mathfrak{d}^{-1} a \bullet \mathfrak{d}^{-1} b)) \\ &= \mathfrak{d}(([\mathbb{S}^{m-1}] * \mathfrak{d}^{-1} a) \bullet ([\mathbb{S}^{m-1}] * \mathfrak{d}^{-1} b)) \\ &= \mathfrak{d}(\mathfrak{d}^{-1}([\mathbb{S}^{m-1}] * a) \bullet \mathfrak{d}^{-1}([\mathbb{S}^{m-1}] * b)) \\ &= j(a) \circ j(b) \end{aligned}$$

using Proposition 2.6.13 and Proposition 2.6.15. □

Remark 2.6.24. This isomorphism is dual to the isomorphism of the cohomology rings of the complements induced by inclusion.

Proposition 2.6.25. Let $X \subset \mathbb{S}^{m-1}$, $Y \subset \mathbb{S}^{n-1}$. Let $q_X \in \mathbb{S}^{m-1} \backslash X$, $q_Y \in \mathbb{S}^{n-1} \backslash Y$ and define Alexander duality in $(\mathbb{S}^{m-1}, X) * (\mathbb{S}^{n-1}, Y) = (\mathbb{S}^{m+n-1}, X * \mathbb{S}^{n-1} \cup \mathbb{S}^{m-1} * Y)$ via a $q \in \{q_X\} * \{q_Y\}$. Orient $\mathbb{S}^{m+n-1} = \mathbb{S}^{m-1} * \mathbb{S}^{n-1}$ by $\epsilon([\mathbb{S}^{m-1}] * [\mathbb{S}^{n-1}])$, $\epsilon \in \{+1, -1\}$. Then the diagram

$$\begin{tikzcd} \tilde{H}_*(X) \otimes \tilde{H}_*(Y) \arrow[r, "(*[\mathbb{S}^{n-1}])\otimes([\mathbb{S}^{m-1}]*)"] & \tilde{H}_*(X * \mathbb{S}^{n-1}) \otimes \tilde{H}_*(\mathbb{S}^{m-1} * Y) \arrow[dd, "\circ"] \\ H_*(\mathbb{S}^{m-1} \backslash \{q_X\}, X) \otimes H_*(\mathbb{S}^{n-1} \backslash \{q_Y\}, Y) \arrow[u, "\cong"', "(-1)^m \mathfrak{d} \otimes \mathfrak{d}"] \arrow[d, "\epsilon \cdot *"] & \\ H_*((\mathbb{S}^{m-1} \backslash \{q_X\}, X) * (\mathbb{S}^{n-1} \backslash \{q_Y\}, Y)) \arrow[r, "\mathfrak{d}"] & \tilde{H}_*(X * \mathbb{S}^{n-1} \cup \mathbb{S}^{m-1} * Y) \end{tikzcd}$$

commutes.

Proof. Let $a \in H_*(\mathbb{S}^{m-1} \backslash \{q_X\}, X)$, $b \in H_*(\mathbb{S}^{n-1} \backslash \{q_Y\}, Y)$. Then

$$\begin{aligned} \epsilon \cdot \mathfrak{d}(a * b) &= \mathfrak{d}((a * [\mathbb{S}^{n-1}]) \bullet ([\mathbb{S}^{m-1}] * b)) \\ &= \mathfrak{d}(a * [\mathbb{S}^{n-1}]) \circ \mathfrak{d}([\mathbb{S}^{m-1}] * b) \\ &= (-1)^m (\mathfrak{d} a * [\mathbb{S}^{n-1}]) \circ ([\mathbb{S}^{m-1}] * \mathfrak{d} b) \end{aligned}$$

by Proposition 2.6.15 and Proposition 2.6.20. □

2.7 Products of classes satisfying the codimension condition

We now turn to our main objective: For an arrangement $\mathcal{A}$ with intersection poset P, compare the combinatorial linking product on $R = \bigoplus_{u \in P} \tilde{H}\big(\Delta(P_{<u})\big)$ with the linking product on $\tilde{H}_*(L_{\mathcal{A}})$ via the isomorphisms of $\mathbb{Z}$-graded abelian groups $\Psi^x \colon R \to \tilde{H}_*(L_{\mathcal{A}})$.

This section deals with products $\Psi^x(a) \circ \Psi^x(b)$ for classes $a \in \tilde{H}\big(\Delta(P_{<u})\big)$ and $b \in \tilde{H}\big(\Delta(P_{<v})\big)$ belonging to pairs u, v that satisfy the codimension condition.

2.7.1 Strategy

We will give a brief description of the idea behind the material in this section. If $u + v = \mathbb{R}^n$ and we consider the $(n - \dim v)$-dimensional arrangement $\mathcal{A}_u = \{w/(u \cap v) \colon w \supset u\}$ and the $(n - \dim u)$-dimensional arrangement $\mathcal{A}_v = \{z/(u \cap v) \colon z \supset v\}$, we will be able to find an inclusion

$$(u \cap v) \times \Big(v/(u \cap v), \bigcup \mathcal{A}_v\Big) \times \Big(u/(u \cap v), \bigcup \mathcal{A}_u\Big) \to \Big(\mathbb{R}^n, \bigcup \mathcal{A}\Big)$$

respectively

$$\mathbb{S}_{u \cap v} * \big(\mathbb{S}_{v/(u \cap v)}, L_{\mathcal{A}_v}\big) * \big(\mathbb{S}_{u/(u \cap v)}, L_{\mathcal{A}_u}\big) \to \big(\mathbb{S}^{n-1}, L_{\mathcal{A}}\big).$$

Therefore we essentially deal with an exterior product, the description of which should be feasible.

2.7.2 The signs $\epsilon_{u,v}$

Let us first get two working descriptions of our signs $\epsilon_{u,v}$ for an arrangement in $\mathbb{R}^n$ with intersection poset P.

Proposition 2.7.1. Let $u, v \in P$, $u + v = \mathbb{R}^n$. Let z be a subspace of $\mathbb{R}^n$ complementary to $u \cap v$. Let $u' = u \cap z$ and $v' = v \cap z$. Identify u with $(u \cap v) \times u'$, v with $(u \cap v) \times v'$, $\mathbb{R}^n$ with $(u \cap v) \times v' \times u'$, and the corresponding spheres accordingly. Now if u' and v' and the corresponding spheres are oriented such that under the described identifications $[\mathbb{S}_u] = [\mathbb{S}_{u \cap v}] * [\mathbb{S}_{u'}]$ and $[\mathbb{S}_v] = [\mathbb{S}_{u \cap v}] * [\mathbb{S}_{v'}]$, then $[\mathbb{S}^{n-1}] = \epsilon_{u,v}[\mathbb{S}_{u,v}] * [\mathbb{S}_{v'}] * [\mathbb{S}_{u'}]$.

Proof. The analogous statement about orientations of disks and their products would be just the definition of ϵ in Section 2.2, and the two are connected by Proposition 2.6.9. $\square$

Proposition 2.7.2. Let $u, v \in P$, $u + v = \mathbb{R}^n$. Then $[\mathbb{S}_u] \bullet [\mathbb{S}_v] = \epsilon_{u,v}[\mathbb{S}_{u \cap v}]$. Here $[\mathbb{S}_0]$ is assumed to be the positive generator of $\tilde{H}_{-1}(\emptyset)$.

Proof. Let the notation be that of the preceding proposition and denote the positive generator of $\tilde{H}_{-1}(\emptyset)$ by $\emptyset$. Then

$$\begin{aligned}[\mathbb{S}_u] \bullet [\mathbb{S}_v] &= \epsilon_{u,v}\big([\mathbb{S}_{u\cap v}] * \emptyset * [\mathbb{S}_{u'}]\big) \bullet \big([\mathbb{S}_{u\cap v}] * [\mathbb{S}_{v'}] * \emptyset\big) = \\ &= \epsilon_{u,v}\big([\mathbb{S}_{u\cap v}] \bullet [\mathbb{S}_{u\cap v}]\big) * \big(\emptyset \bullet [\mathbb{S}_{v'}]\big) * \big([\mathbb{S}_{u'}] \bullet \emptyset\big) = \\ &= \epsilon_{u,v}[\mathbb{S}_{u\cap v}] * \emptyset * \emptyset = \epsilon_{u,v}[\mathbb{S}_{u\cap v}]\end{aligned}$$

by Proposition 2.6.15 and Proposition 2.6.13. □

2.7.3 Geometrical description of the linking product in a link

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$, $u, v \in P$ and $u + v = \mathbb{R}^n$, $u \cap v = 0$. In this case we can identify $\mathbb{S}^{n-1}$ with $\mathbb{S}_v * \mathbb{S}_u$. Let $L_u^v := \mathbb{S}_v \cap \bigcup\{w \in \mathcal{A} \colon w < u\}$. Note that $L_u^v * \mathbb{S}_u \subset L_{\mathcal{A}}$. Define

$$\phi_u^v \colon \Delta(P_{<u}) \to L_u^v$$

like ϕ_u earlier, but with

$$\phi_u^v(w) = x_{v\cap w}.$$

Now choose generic points x'_w, such that $x_w \in x'_{w\cap v} * \mathbb{S}_u$ for all $w < u$ and $\mathbb{S}_v * x_w \in x'_{w\cap u}$ for all $w < v$. Comparing the subarrangement of $\mathcal{A}$ consisting of all subspaces containing v and the arrangement in u given by the intersections with u of the former subspaces, we are just in the situation of Proposition 2.3.2. Therefore, for $b \in \tilde{H}_*(\Delta(P_{<v}))$ we have

$$\Psi^x(b) = \Phi_*^x([\mathbb{S}^{\dim(v)-1}] * b) = [\mathbb{S}_v] * ({\phi'}_v^u)_*(b)$$

and for $a \in \tilde{H}_*(\Delta(P_{<u}))$

$$\Psi^x(a) = \Phi_*^x([\mathbb{S}^{\dim(u)-1}] * a) = (-1)^{(\dim u)(|a|+1)}({\phi'}_u^v)_*(a) * [\mathbb{S}_u],$$

where the sign comes from Proposition 2.6.8.

Now, using Proposition 2.6.25 we get

$$\begin{aligned}&\big((-1)^{(\dim u)(|a|+1)}({\phi'}_u^v)_*(a) * [\mathbb{S}_u]\big) \circ \big([\mathbb{S}_v] * ({\phi'}_v^u)_*(b)\big) = \\ &\qquad = \epsilon_{u,v} \cdot (-1)^{\dim v + (\dim u)(|a|+1)} \mathfrak{d}\big(\mathfrak{d}^{-1}({\phi'}_u^v)_*(a) * \mathfrak{d}^{-1}({\phi'}_v^u)_*(b)\big),\end{aligned}$$

since the ϵ in Proposition 2.6.25 has been identified as $\epsilon_{u,v}$ in Proposition 2.7.1. We summarize:

Proposition 2.7.3. Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$, $u, v \in P$ and $u + v = \mathbb{R}^n$, $u \cap v = 0$. There is a choice of generic points $\{x_w\}$ such that for all $a \in \tilde{H}_*(\Delta(P_{<u}))$ and $b \in \tilde{H}_*(\Delta(P_{<v}))$ we have

$$\Psi^x(a) \circ \Psi^x(b) = \epsilon_{u,v} \cdot (-1)^{\dim v + (\dim u)(|a|+1)} \mathfrak{d}\big(\mathfrak{d}^{-1}(\phi_u^v)_*(a) * \mathfrak{d}^{-1}(\phi_v^u)_*(b)\big).$$

Here, $\mathbb{S}^{n-1}$ is identified with $\mathbb{S}_v * \mathbb{S}_u$. Given an arbitrary choice of generic points $\{x'_w\}$, x can be chosen in such a way that Ψ^x and $\Psi^{x'}$ agree on $\tilde{H}_*(\Delta(P_{<u}))$ and $\tilde{H}_*(\Delta(P_{<v}))$. □

When compared with the expression in Remark 2.5.2 this already looks promising.

2.7.4 Combinatorial description of the linking product.

Now that we have a nice geometrical description of the product of two classes satisfying the codimension condition, we are ready to describe it in terms of the combinatorial data. The rest of this section will be devoted to proving the following

Proposition 2.7.4. Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$, $u, v \in P$ and $u+v = \mathbb{R}^n$. There is a choice of generic points $\{x_w\}$ such that for all $a \in \tilde{H}_*(\Delta(P_{<u}))$ and $b \in \tilde{H}_*(\Delta(P_{<v}))$ we have

$$\Psi^x(a) \circ \Psi^x(b) = \Psi^x(a \circ b),$$

where the product on the right hand side refers to the combinatorial one given in Definition 2.5.1. For an arbitrary choice of generic points $\{x'_w\}$, x can be chosen in such a way that Ψ^x and $\Psi^{x'}$ agree on $\tilde{H}_*(\Delta(P_{<u}))$ and $b \in \tilde{H}_*(\Delta(P_{<v}))$.

Caution 2.7.5. The generic points $\{x_w\}$ above depend on u and v.

Lemma 2.7.6. If Proposition 2.7.4 is true for all arrangements $\mathcal{A}$ and all u, v with $u \cap v = 0$, then it is true in general.

Proof. Because of the naturality of the linking product with respect to inclusions, we can calculate it in the subarrangement of $\mathcal{A}$ consisting of all subspaces containing $u \cap v$. This in turn gives rise to an arrangement $\mathcal{A}'$ by considering the intersections with a subspace of $\mathbb{R}^n$ complementary to $u \cap v$, which is related to $\mathcal{A}$ in the manner described in Proposition 2.3.2 (with the roles of $\mathcal{A}$ and $\mathcal{A}'$ exchanged). Combining this with Proposition 2.6.23 we get

$$\Psi^x(a \circ b) = [\mathbb{S}_{u\cap v}] * \Psi^{x'}(a' \circ b'),$$

where a, b can be identified with a', b' and the primes are only needed to distinguish the combinatorial $\circ$ in the different rings, and

$$\begin{aligned}\Psi^x(a) \circ \Psi^x(b) &= \Big([\mathbb{S}_{u\cap v}] * \Psi^{x'}(a')\Big) \circ \Big([\mathbb{S}_{u\cap v}] * \Psi^{x'}(b')\Big) = \\ &= (-1)^{\dim(u\cap v)}[\mathbb{S}_{u\cap v}] * \Big(\Psi^{x'}(a') \circ \Psi^{x'}(b')\Big).\end{aligned}$$

Thus the sign in the definition of $a \circ b$ should be $(-1)^{\dim(u\cap v)}$ times the sign in $a' \circ b'$, which is the case if $n = n' + \dim(u \cap v)$, $n - \dim v = n' - \dim v'$, $\epsilon_{u,v} = \epsilon'_{u',v'}$. Only the latter needs to be checked, and we have

$$\begin{aligned}\epsilon_{u,v}[\mathbb{S}_{u\cap v}] = [\mathbb{S}_u] \bullet [\mathbb{S}_v] &= ([\mathbb{S}_{u\cap v}] * [\mathbb{S}_{u'}]) \bullet ([\mathbb{S}_{u\cap v}] * [\mathbb{S}_{v'}]) = \\ &= ([\mathbb{S}_{u\cap v}] \bullet [\mathbb{S}_{u\cap v}]) * ([\mathbb{S}_{u'}] \bullet [\mathbb{S}_{v'}]) = \epsilon'_{u',v'}[\mathbb{S}_{u\cap v}],\end{aligned}$$

using Proposition 2.7.2. □

Proposition 2.7.7. In the situation of Proposition 2.7.3 and with the terminology of Section 2.5 we have

$$\mathfrak{d}\left(\mathfrak{d}^{-1}\phi_u^v(a) * \mathfrak{d}^{-1}\phi_v^u(b)\right) = \vee_*(\lambda(a \otimes b)).$$

Proof. Denote the point of $\mathbb{S}^{n-1}\backslash L_{\mathcal{A}}$ used to define Alexander duality by q. Assume that points $q_u \in \mathbb{S}_u\backslash L_v^u$, $q_v \in \mathbb{S}_v\backslash L_u^v$ are chosen such that q is halfway between q_v and q_u. q will have to be moved to achieve this, but this does not matter, since Alexander duality depends only on the connected component of $\mathbb{S}^{n-1}\backslash L_{\mathcal{A}}$ in which q lies. Set $x_u = -q_u$, $x_v = -q_v$. Additionally to the x_w chosen for $u \in P_{<0}$, set $x_0 = 0$, $x_\perp = -q$ and extend ϕ_0, ϕ_u^v, ϕ_v^u accordingly (1-simplices involving x_0 will of course be mapped to straight lines, all others still to geodesics). In Proposition 2.7.3 we have already chosen x such that $x_{w\vee z} \in \{x_{w\vee v}\} * \{x_{u\vee z}\}$ for all $w \in P_{<u}$ and $z \in P_{<v}$. Assume that they are indeed at the center of that line and note that this now holds for all $w \in \hat{P}_{\leq u}$ and $z \in \hat{P}_{\leq v}$. Now consider the diagram in Figure 2.1 on page 49. The diagram is commutative: α is essentially Proposition 2.6.10, β commutes because of our choice of $\phi_u^v(\perp) = x_v = -q_v$, $\phi_v^u(\perp) = x_u = -q_u$. γ commutes because of our choice of $x_{w\cap z}$ above. Commutativity of δ and ε is due to $\phi_u^v(u) = \phi_v^u(v) = x_0$. Now the composition of the arrow on top with the arrows on the extreme right is the left hand side of the equation to be proved, while the arrows on the left together with the bottom arrow yield the right hand side. □

Proof of Proposition 2.7.4. Because of Lemma 2.7.6 we may assume $u \cap v = 0$, thus we just have to show that combining Proposition 2.7.3 with Proposition 2.7.7 yields the correct sign, and indeed

$$(-1)^{\dim v + (\dim u)(|a|+1)} = (-1)^{n+|a|(n-\dim v)},$$

since $\dim u + \dim v = n$. □

2.8 Products of classes not satisfying the codimension condition

Assuming that the arrangement $\mathcal{A}$ does not contain real hyperplanes we show that for any pair $u, v \in P$ that does not satisfy the codimension condition there is pair of model maps leading to trivial multiplication.

Proposition 2.8.1. Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{R}^n$ not containing real hyperplanes with intersection poset P. Let $u, v \in P$ be two subspaces that do not satisfy the codimension condition, i.e., $u + v \neq \mathbb{R}^n$. Then there is a pair of model maps $\Phi^x, \Phi^{x'} : \mathcal{L}_{\mathcal{A}} \longrightarrow L_{\mathcal{A}}$ such that the following composition

$$
\begin{tikzcd}
 & \tilde H_r(P_{<u})\otimes \tilde H_s(P_{<v}) \arrow[r, "(\phi^v_u)_*\otimes(\phi^u_v)_*"] \arrow[d, "C_\perp\otimes C_\perp"] \arrow[ddd, bend right, "C_{u\perp}\otimes C_{v\perp}"'] \arrow[rdd, phantom, "\beta"] & \tilde H_r(L^v_u)\otimes \tilde H_s(L^u_v) \arrow[dd, "\mathfrak{d}^{-1}\otimes\mathfrak{d}^{-1}"] \\
 & H_{r+1}(\hat P_{<u},P_{<u})\otimes H_{s+1}(\hat P_{<v},P_{<v}) \arrow[rd, "(\phi^v_u)_*\otimes(\phi^u_v)_*"] \arrow[dd, "C_u\otimes C_v"'] & \\
 & & H_{r+1}(\mathbb{S}_v\setminus\{q_v\},L^v_u)\otimes H_{s+1}(\mathbb{S}_u\setminus\{q_u\},L^u_v) \arrow[dd, "(x_0*)\otimes(x_0*)"] \arrow[dddddddd, bend left, "*"] \arrow[ld, phantom, "\delta"] \\
 & H_{r+2}(\hat P_{\le u},\hat P_{<u}\cup P_{\le u})\otimes H_{s+2}(\hat P_{\le v},\hat P_{<v}\cup P_{\le v}) \arrow[rd, "(\phi^v_u)_*\otimes(\phi^u_v)_*"] \arrow[dd, "\times"'] & \\
 & & H_{r+2}(\mathbb{D}_v,\mathbb{S}_v\cup 0*L^v_u)\otimes H_{s+2}(\mathbb{D}_u,\mathbb{S}_u\cup 0*L^u_v) \arrow[dd, "\times"] \\
 & H_{r+s+4}\Big(\hat P_{\le u}\times\hat P_{\le v},(\hat P_{\le u}\times\hat P_{\le v})_{<(u,v)}\cup(\hat P_{\le u}\times\hat P_{\le v})_{>\perp}\Big) \arrow[rd, "(\phi^v_u\times\phi^u_v)_*"] \arrow[dd, "C^{-1}_{(u,v)}"'] \arrow[ddddl, bend right, "C^{-1}_{(u,v)\perp}"'] & {} \arrow[r, phantom, "\alpha"] & {} \\
 & & H_{r+s+4}(\mathbb{D}_v\times\mathbb{D}_u,\mathbb{D}_v\times\mathbb{S}_u\cup\mathbb{S}_v\times\mathbb{D}_u\cup 0*(\mathbb{D}_v\times L^u_v\cup L^v_u\times\mathbb{D}_u)) \arrow[dd, "(x_0*)^{-1}"] \\
 & H_{r+s+3}\left((\hat P_{\le u}\times\hat P_{\le v})_{<(u,v)},(\hat P_{\le u}\times\hat P_{\le v})_{\substack{<(u,v)\\ >\perp}}\right) \arrow[rd, "(\phi^v_u\times\phi^u_v)_*"] \arrow[ddl, "C_\perp^{-1}"'] \arrow[ddd, "\vee_*"] \arrow[ru, phantom, "\varepsilon"] & \\
 & & H_{r+s+3}(\mathbb{D}_v\times\mathbb{S}_u\cup\mathbb{S}_v\times\mathbb{D}_u,\mathbb{D}_v\times L^u_v\cup L^v_u\times\mathbb{D}_u) \arrow[dd] \\
\tilde H_{r+s+2}\left((\hat P_{\le u}\times\hat P_{\le v})_{\substack{<(u,v)\\ >\perp}}\right) \arrow[ddr, "\vee_*"'] & & \\
 & H_{r+s+3}(\hat P_{<0},P_{<0}) \arrow[r, "(\phi_0)_*"] \arrow[d, "\mathfrak{d}"] \arrow[ru, phantom, "\gamma"] & H_{r+s+3}(\mathbb{S}_v*\mathbb{S}_u,\mathbb{S}_v*L^u_v\cup L^v_u*\mathbb{S}_u) \arrow[d, "\mathfrak{d}"] \\
 & \tilde H_{r+s+2}(P_{<0}) \arrow[r, "(\phi_0)_*"] \arrow[d, equal] & \tilde H_{r+s+2}(\mathbb{S}_v*L^u_v\cup L^v_u*\mathbb{S}_u) \arrow[d] \\
 & \tilde H_{r+s+2}(P_{<0}) \arrow[r, "\Psi^{x'}"] & \tilde H_{r+s+2}(L_{\mathcal{A}})
\end{tikzcd}
$$

Figure 2.1: Diagram needed in the proof of Proposition 2.7.7.

is the zero map.

$$\tilde{H}_r(\mathbb{S}^{\dim u-1} * \Delta(P_{<u})) \otimes \tilde{H}_s(\mathbb{S}^{\dim v-1} * \Delta(P_{<v})) \xrightarrow{\Phi_*^x \otimes \Phi_*^{x'}} \tilde{H}_r(L_{\mathcal{A}}) \otimes \tilde{H}_s(L_{\mathcal{A}}) \xrightarrow{\circ} \tilde{H}_{r+s+2-n}(L_{\mathcal{A}}). \tag{2.1}$$

Before proving Proposition 2.8.1 we will spend some time on thickening the arrangement, i.e., replacing all subspaces by a thickened copy, to have space for further constructions.

Definition 2.8.2. Let $\mathcal{A}$ be a subspace arrangement in $\mathbb{R}^n$, $\vec{n} \in \mathbb{R}^n$ a unit vector. For $\xi \in \mathbb{R}$ we define the *slanted arrangement in direction* $\vec{n}$

$$\mathcal{A}(\vec{n},\xi) = \{a + \xi \cdot \vec{n} : a \in \mathcal{A}\},$$

and for $\varepsilon > 0$ we define the *thickened arrangement in direction* $\vec{n}$ using the same notation

$$\mathcal{A}(\vec{n},[-\varepsilon,+\varepsilon]) = \{a + [-\varepsilon,+\varepsilon] \cdot \vec{n} : a \in \mathcal{A}\}.$$

Proposition 2.8.3. For sufficiently small $\varepsilon > 0$ the following inclusions are homotopy equivalences for every $\xi \in [-\varepsilon,+\varepsilon]$:

$$\mathbb{S}^{n-1} \cap \bigcup_{a \in \mathcal{A}(\vec{n},\xi)} a \hookrightarrow \mathbb{S}^{n-1} \cap \bigcup_{\mathbf{a} \in \mathcal{A}(\vec{n},[-\varepsilon,+\varepsilon])} \mathbf{a}\,,$$
$$\mathbb{S}^{n-1} \setminus \bigcup_{\mathbf{a} \in \mathcal{A}(\vec{n},[-\varepsilon,+\varepsilon])} \mathbf{a} \hookrightarrow \mathbb{S}^{n-1} \setminus \bigcup_{a \in \mathcal{A}(\vec{n},\xi)} a\,.$$

Now consider the following orientation preserving homeomorphism $\gamma_\xi : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ of the pairs $(\mathbb{S}^{n-1}, L_{\mathcal{A}})$ and $(\mathbb{S}^{n-1}, \mathbb{S}^{n-1} \cap \bigcup_{a \in \mathcal{A}(\vec{n},\xi)} a)$ which is just travelling along a geodesic from $-\vec{n}$ to $\vec{n}$ the right amount of time. In formulae:

$$x \in L_{\mathcal{A}} \stackrel{\gamma_\xi}{\longmapsto} t \cdot x + \xi \cdot \vec{n},$$

where $t \geq 0$ is such that $\|t \cdot x + \xi \cdot \vec{n}\| = 1$. Then the following composition of maps yields a homotopy equivalence for every $\xi \in [-\varepsilon,+\varepsilon]$ and any two such are homotopic:

$$L_{\mathcal{A}} \xrightarrow{\gamma_\xi} \mathbb{S}^{n-1} \cap \bigcup_{a \in \mathcal{A}(\vec{n},\xi)} a \hookrightarrow \mathbb{S}^{n-1} \cap \bigcup_{\mathbf{a} \in \mathcal{A}(\vec{n},[-\varepsilon,+\varepsilon])} \mathbf{a}.$$

Using the same construction we also obtain a homotopy equivalence

$$M_{\mathcal{A}} \xrightarrow{\simeq} \mathbb{S}^{n-1} \setminus \bigcup_{a \in \mathcal{A}(\vec{n},\xi)} a.$$

The homotopy equivalences are such that the following diagram commutes

$$\begin{array}{ccccc}
\tilde{H}_*(L_{\mathcal{A}}) & \xrightarrow{\cong} & \tilde{H}_*(\mathbb{S}^{n-1} \cap \bigcup_{a\in\mathcal{A}(\vec{n},\xi)} a) & \xrightarrow{\cong} & \tilde{H}_*(\mathbb{S}^{n-1} \cap \bigcup_{\mathbf{a}\in\mathcal{A}(\vec{n},[-\varepsilon,+\varepsilon])} \mathbf{a}) \\
\downarrow{\scriptstyle AD_q} & & \downarrow{\scriptstyle AD_{q'}} & & \downarrow{\scriptstyle AD_{q'}} \\
H^*(M_{\mathcal{A}},\{q\}) & \xleftarrow{\cong} & H^*(\mathbb{S}^{n-1} \setminus \bigcup_{a\in\mathcal{A}(\vec{n},\xi)} a,\{q'\}) & \xrightarrow{\cong} & H^*(\mathbb{S}^{n-1} \setminus \bigcup_{\mathbf{a}\in\mathcal{A}(\vec{n},[-\varepsilon,+\varepsilon])} \mathbf{a},\{q'\}),
\end{array}$$

where $q' = \gamma_\xi(q)$. □

Proof of Proposition 2.8.1. The construction of the maps Φ^x and $\Phi^{x'}$ splits up into two parts. One is concerned with the order complex maps ϕ_u, $u \in P$, whereas the other one is concerned with the sphere portion.

The order complex portion

If the codimension condition is not satisfied there certainly is a hyperplane $H \subset \mathbb{R}^n$ containing $u + v$. Let $\vec{n}$ be a unit normal vector to H. Construct two model maps Φ^x and $\Phi^{x'}$ by choosing the generic points in such a way that the images of $\Delta(P_{<u})$ under Φ^x, respectively $\Delta(P_{<v})$ under $\Phi^{x'}$, are situated on different sides of H, say

$$\langle \vec{n}, \Phi^x(\Delta(P_{<u}))\rangle \geq 0\,,$$
$$\langle \vec{n}, \Phi^{x'}(\Delta(P_{<v}))\rangle \leq 0\,.$$

The sphere portion

If two spaces $u, v \in P$ have non–empty intersection there is no model map given by $\iota_u * \phi_u$, $u \in P$, such that the images of $\iota_u : \mathbb{S}^{\dim(u)-1} \to L_{\mathcal{A}}$ and $\iota_v : \mathbb{S}^{\dim(v)-1} \to L_{\mathcal{A}}$ are disjoint. Therefore we thicken the arrangement. By Proposition 2.8.3 we can compute the linking product in the space $\mathbb{S}^{n-1} \cap \bigcup_{\mathbf{a}\in\mathcal{A}(\vec{n},[-\varepsilon,+\varepsilon])} \mathbf{a}$ for a suitable choice of $\varepsilon > 0$. We consider the compositions

$$\mathcal{L}_{\mathcal{A}} \xrightarrow{\Phi^x} L_{\mathcal{A}} \xrightarrow{\gamma_{+\varepsilon}} \mathbb{S}^{n-1} \cap \bigcup_{a\in\mathcal{A}(\vec{n},+\varepsilon)} a \hookrightarrow \mathbb{S}^{n-1} \cap \bigcup_{\mathbf{a}\in\mathcal{A}(\vec{n},[-\varepsilon,+\varepsilon])} \mathbf{a}\ , \text{ respectively}$$
$$\mathcal{L}_{\mathcal{A}} \xrightarrow{\Phi^{x'}} L_{\mathcal{A}} \xrightarrow{\gamma_{-\varepsilon}} \mathbb{S}^{n-1} \cap \bigcup_{a\in\mathcal{A}(\vec{n},-\varepsilon)} a \hookrightarrow \mathbb{S}^{n-1} \cap \bigcup_{\mathbf{a}\in\mathcal{A}(\vec{n},[-\varepsilon,+\varepsilon])} \mathbf{a}.$$

Consider the images of $\mathbb{S}^{\dim(u)-1} * \Delta(P_{<u})$ under the first map, respectively of $\mathbb{S}^{\dim(v)-1} * \Delta(P_{<v})$ under the second. They are separated in the softened link $\mathbb{S}^{n-1} \cap \bigcup_{\mathbf{a}\in\mathcal{A}(\vec{n},\varepsilon)} \mathbf{a}$ by the hyperplane H. Furthermore the dimension of these images is at most $n - 3$, since $\mathcal{A}$ does not contain any hyperplanes. Hence by Corollary 2.6.22 the composition of maps (2.1) is the zero map. □

Corollary 2.8.4. If $\mathcal{A}$ does not contain real hyperplanes and $u, v \in P$ do not satisfy the codimension condition then there is a pair of Goresky-MacPherson isomorphisms $\Psi^x, \Psi^{x'}$ such that the following composition is the zero map.

$$\begin{aligned} \tilde{H}_r(\Delta(P_{<u})) \otimes \tilde{H}_s(\Delta(P_{<v})) &\xrightarrow{\Psi^x \otimes \Psi^{x'}} \tilde{H}_r(L_{\mathcal{A}}) \otimes \tilde{H}_s(L_{\mathcal{A}}) \\ &\xrightarrow{\quad \circ \quad} \tilde{H}_{r+s+2-n}(L_{\mathcal{A}}). \end{aligned}$$

□

2.9 On (≥ 2)-arrangements

In this section we will show that (≥ 2)-arrangements are very friendly. To be more precise we will show that there is a universal description of the multiplication in the cohomology based on just one model map. This is possible since for (≥ 2)-arrangements any two model maps are homotopic.

2.9.1 Invariance of model maps

Lemma 2.9.1. Let $\mathcal{A}$ be a (≥ 2)-arrangement, and let $\Phi^x, \Phi^{x'} : \mathcal{L}_{\mathcal{A}} \to L_{\mathcal{A}}$ be two model maps. Assume that the choice of generic points x and x' coincide except for the choice of x_v respectively x'_v. Then Φ_x and $\Phi_{x'}$ are homotopic.

Proof. We want to apply Lemma 2.3.1, i.e., we have to show the existence of a path of generic points in v joining x_v and $x_{v'}$. This is trivial since $x_v, x_{v'} \in \mathbb{S}_v \setminus \bigcup_{u>v} \mathbb{S}_u$ and $\operatorname{codim}(u \subset v) \geq 2$ for all $u > v$. □

As a consequence we obtain the following result.

Proposition 2.9.2. For a (≥ 2)-arrangement the homotopy type of a model map $\Phi^x : \mathcal{L}_{\mathcal{A}} \to L_{\mathcal{A}}$ does not depend on the choice of the generic points. In particular, any two Goresky-MacPherson isomorphisms Ψ^x and $\Psi^{x'}$ coincide. □

2.9.2 Description of the cohomology ring

Theorem 2.9.3. *If $\mathcal{A}$ is a (≥ 2)-arrangement, then the ring structure of the cohomology of the complement of $\mathcal{A}$ is given by the combinatorial data via*

$$\begin{aligned} \tilde{H}_r(\Delta(P_{<u})) \otimes \tilde{H}_s(\Delta(P_{<v})) &\longrightarrow \tilde{H}_{r+s+2}(\Delta(P_{<u\cap v})) \\ a \otimes b &\longmapsto a \circ b. \end{aligned}$$

Proof. Proposition 2.9.2 implies in particular that any two model maps are homologically the same. The theorem now follows from Proposition 2.7.4 if the codimension condition is satisfied and from Corollary 2.8.4 in the other case. □

2.9.3 Geometric (≥ 2)-arrangements

Historical remarks

In the case of a complex hyperplane arrangement a description of the integral cohomology ring in terms of generators and relations was given by Arnol'd [Arn69] and Brieskorn [Bri73]. A combinatorial description of their result was given by Orlik and Solomon [OS80].

Goresky and MacPherson [GM88, Chapter III] computed the Poincaré polynomial for the class of c-arrangements which can be considered as generalized hyperplane arrangements.

Definition 2.9.4. An arrangement $\mathcal{A} = \{H_1, \ldots, H_m\}$ in $\mathbb{R}^n$ is called a *c-arrangement,* if $\operatorname{codim}_{\mathbb{R}}(H_i) = c$ for all i, and for all pairs of elements $u \subset v$ in the intersection poset P we have that $\operatorname{codim}_{\mathbb{R}}(u \subset v)$ is an integral multiple of c.

Ziegler has given a presentation of the integral cohomology of the complement of 2-arrangements [Zie93] generalizing the result of Orlik and Solomon. He used this presentation to give an example of a complex (in the sense of Gauß) and a 2-arrangement with the same combinatorial data – intersection poset and dimension function – which yield different cohomology rings. In his article the spaces are implicitly oriented by the choice of normal vectors for pairs of real hyperplanes whose intersections yield the codimension-2-spaces in the arrangement.

Hyperplane- and c-arrangements belong to the much larger class of arrangements with geometric intersection lattice.

Definition 2.9.5. A lattice L is *geometric*, if the following conditions hold:

- ▷ it is *atomic,* i.e., every element is the join of a set of atoms,
- ▷ it admits a *rank function* rk, i.e., for all $x \in L$ all maximal chains from the smallest element to x have the same length (which is $\operatorname{rk}(x) + 1$),
- ▷ its rank function is *semimodular*, i.e., for all $x, y \in L$ we have $rk(x \vee y) + \operatorname{rk}(x \wedge y) \leq \operatorname{rk}(x) + \operatorname{rk}(y)$.

For the class of complex arrangements with geometric intersection lattice Feichtner and Ziegler [FZ00] and Yuzvinsky [Yuz98], [Yuz99] have given a presentation of the integral cohomology ring of the complement.

A presentation for the cohomology ring of a geometric (≥ 2)-arrangement

We give a presentation for the cohomology ring of the complement of a (≥ 2)-arrangement with geometric intersection lattice, thus generalizing the results in [FZ00], [Yuz98] respectively. Since the proof of our theorem is along the lines of Yuzvinsky's article [Yuz99] applying our Theorem 2.9.3, we omit the details of the proof.

Recall that a subset σ of atoms of a geometric lattice L is called *independent* if $\operatorname{rk} \bigvee \sigma = |\sigma|$, otherwise it is called *dependent*.

Theorem 2.9.6. *Let $\mathcal{A}$ be a (≥ 2)-arrangement in $\mathbb{R}^n$ with geometric intersection lattice L. Fix an arbitrary linear order on the set of atoms of L. Then the integral cohomology ring of the complement $M_{\mathcal{A}}$ has the following presentation.*

$$0 \longrightarrow I \longrightarrow T^* \left(\bigoplus_{\sigma \text{ ind.}} \mathbb{Z} \cdot e_\sigma \right) \stackrel{\pi}{\longrightarrow} H^*(M_{\mathcal{A}};\mathbb{Z}) \longrightarrow 0,$$

where T^* denotes the tensor algebra and the sum in the middle is over all independent sets σ of atoms of L, and $\pi(e_\sigma) \in H^{(n-\dim(\cap\sigma))-|\sigma|}(M_{\mathcal{A}};\mathbb{Z})$. The ideal I of relations is generated by the following three families of elements.

(i) For every minimal dependent set $\sigma = \{a_0, \ldots, a_k\}$ of atoms of L:

$$\sum_{i=0}^{k} (-1)^i e_{\sigma \setminus \{a_i\}}.$$

(ii) For all pairs σ, τ of independent sets of atoms of L such that $\dim(\bigcap\sigma \cap \bigcap\tau) = \dim(\bigcap\sigma) + \dim(\bigcap\tau) - n$:

$$e_\sigma \wedge e_\tau - \epsilon_{\cap\sigma,\cap\tau}(-1)^{n+(|\sigma|-1)(n-\dim\cap\tau)}(-1)^{\operatorname{sign}(\sigma,\tau)} e_{\sigma\cup\tau},$$

where $\operatorname{sign}(\sigma,\tau)$ is the sign of the permutation that orders elements of σ followed by elements of τ ascendingly according to the chosen linear order.

(iii) For all pairs σ, τ of independent sets of atoms of L such that $\dim(\bigcap\sigma \cap \bigcap\tau) \neq \dim(\bigcap\sigma) + \dim(\bigcap\tau) - n$:

$$e_\sigma \wedge e_\tau.$$

In particular we obtain a presentation for general real c-arrangements generalizing Ziegler's result in [Zie93].

Corollary 2.9.7. Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ be a c-arrangement in $\mathbb{R}^n$. Then the integral cohomology ring of the complement $M_{\mathcal{A}}$ has the presentation

$$0 \longrightarrow I \longrightarrow \Lambda^*\mathbb{Z}^m \stackrel{\pi}{\longrightarrow} H^*(M_{\mathcal{A}};\mathbb{Z}) \longrightarrow 0,$$

if c is even, Λ^* denoting the exterior algebra, and

$$0 \longrightarrow I \longrightarrow S^*\mathbb{Z}^m \stackrel{\pi}{\longrightarrow} H^*(M_{\mathcal{A}};\mathbb{Z}) \longrightarrow 0,$$

if c is odd, S^* denoting the symmetric algebra, and $\pi(e_i) \in H^{c-1}(M_{\mathcal{A}};\mathbb{Z})$ for the canonical basis $\{e_1,\ldots,e_m\}$ of $\mathbb{Z}^m$. The ideal I of relations is generated by

$$\sum_{i=0}^{k}(-1)^i \epsilon(a_0,\ldots,\widehat{a_i},\ldots,a_k)e_{a_0}\wedge\cdots\wedge\widehat{e_{a_i}}\wedge\cdots\wedge e_{a_k},$$

for all minimal dependent sets $\{H_{a_0},\ldots,H_{a_k}\}$, where

$$\epsilon(i_0,\ldots,i_r) = \epsilon_{H_{i_0},H_{i_1}}\epsilon_{H_{i_0}\cap H_{i_1},H_{i_2}}\cdots\epsilon_{H_{i_0}\cap H_{i_1}\cap\cdots\cap H_{i_{r-1}},H_{i_r}}$$

for any subset $\{i_0,\ldots,i_r\}\subset\{1,\ldots,m\}$.

2.10 General real arrangements are not as nice as (≥ 2)-arrangements

In this section we show that many results fail if we are not dealing with (≥ 2)-arrangements.

The model map depends on choices - an example

We start with the fact that in general the homotopy class of Φ^x will depend on the choice of the generic points x. In fact, even the isomorphism Φ^x_* depends upon it. We give a rather simple example to demonstrate this. Let $\mathcal{A}$ be the arrangement given by two coordinate hyperplanes in $\mathbb{R}^3$:

$$\mathcal{A} = \{xy\text{-plane}, xz\text{-plane}\}.$$

In Figure 2.2 the homotopy model $\mathcal{L}_{\mathcal{A}}$ is shown, and two possible model maps Φ^x and $\Phi^{x'}$ are sketched. They differ by the choice of the generic point for xz-plane indicated by the large black vertex.
Consider the image of a generating cycle of $\tilde{H}_1(\mathbb{S}^0 * \Delta(P_{<x\text{-axis}}))$ under Φ^x_* (see dashed line in Figure 2.2). Which element of $\tilde{H}_*(\mathcal{L}_{\mathcal{A}})$ maps under $\Phi^{x'}_*$ onto this element? It is the sum of the generators of $\tilde{H}_1(\mathbb{S}^0 * \Delta(P_{<x\text{-axis}}))$ and $\tilde{H}_1(\mathbb{S}^1 * \Delta(P_{<xz\text{-plane}})) = \tilde{H}_1(\mathbb{S}^1)$ with the appropriate signs (compare Figure 2.2). Note that the sign of the second class switches upon changing the orientation of xy-plane. We conclude that the induced maps Φ^x_* and $\Phi^{x'}_*$ are different, and even the induced double gradings differ. We keep in mind:

Note 2.10.1. In general, the homotopy class of a model map $\Phi^x : \mathcal{L}_{\mathcal{A}} \to L_{\mathcal{A}}$ depends on the choice of the generic points x_u, $u \in P$. This is the fact even for the induced isomorphism Φ^x_* in homology, and therefore also for the Goresky-MacPherson isomorphism Ψ^x.

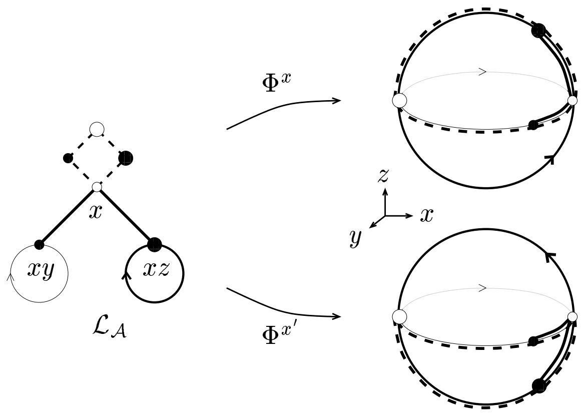


Figure 2.2: Two non–homotopic model maps

2.10.1 The codimension condition is satisfied

Consider a general real subspace arrangement $\mathcal{A}$ and fix a choice x of generic points, respectively a Goresky-MacPherson isomorphism Ψ^x. Let $u, v \in P$ be a pair of spaces satisfying the codimension condition, $a \in \tilde{H}_*(\Delta(P_{<u}))$, and $b \in \tilde{H}_*(\Delta(P_{<v}))$. The linking product of $\Psi^x(a)$ and $\Psi^x(b)$ is in general not equal to $\Psi^x(a \circ b)$. Proposition 2.7.4 only shows that there is a choice of generic points x' such that $\Psi^x(a) \circ \Psi^x(b) = \Psi^{x'}(a \circ b)$.

An example

Consider the arrangement $\mathcal{A}$ consisting of the three coordinate hyperplanes in $\mathbb{R}^3$. The intersection poset P is shown in Figure 2.3.

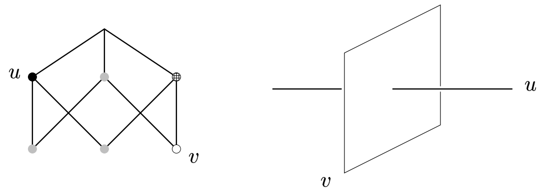


Figure 2.3: The intersection poset of the coordinate hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^3$ and the spaces u and v.

Let u and v be the two spaces shown in Figure 2.3. They satisfy the codimension condition. The bold dashed lines in the two top spheres in Figure 2.4 represent the two homology classes $\Psi^x(a)$ and $\Psi^x(b)$ where $a \in \tilde{H}_*(\Delta(P_{<u}))$ and $b \in \tilde{H}_*(\Delta(P_{<v}))$.
The 2-dimensional classes in the respective relative homologies bounding $\Psi^x(a)$ and $\Psi^x(b)$ are sketched by the funny thin dashed lines. See Proposition 2.6.20

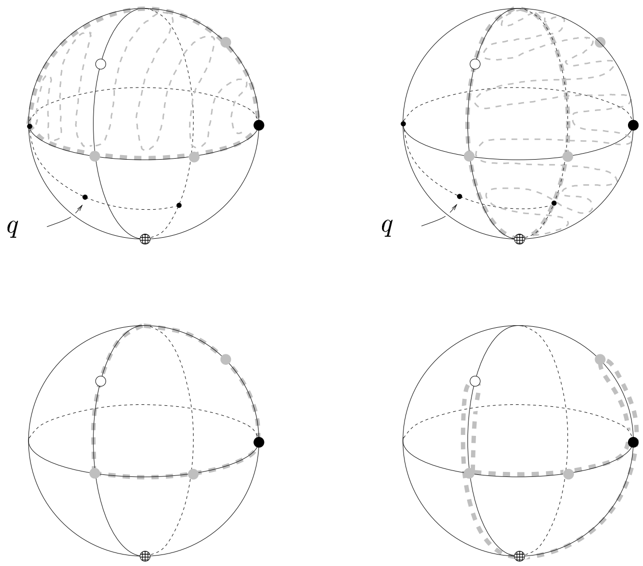


Figure 2.4: In general $\Psi^x(a) \circ \Psi^x(b)$ does not coincide with $\Psi^x(a \circ b)$ for a fixed model map Ψ^x

for details. The boundary of the intersection product of the two 2-dimensional classes is shown by the bold dashed line in the bottom sphere on the left in Figure 2.4. By Proposition 2.6.20 this class yields the linking product of $\Psi^x(a)$ and $\Psi^x(b)$. But this class does not coincide with the class $\Psi^x(a \circ b)$ indicated by the bold dashed line in the bottom sphere on the right in Figure 2.4.

2.10.2 The codimension condition is not satisfied

Even if the codimension is not satisfied in the general case non-trivial muliplication can occur.

A "Hopf-example"

We give an example of an arrangement $\mathcal{A}$ consisting of six 2-dimensional planes in $\mathbb{R}^4$. Consider the "Hopf-arrangement" of six affine lines in $\mathbb{R}^3$ as shown in Figure 2.5.
Let the arrangement $\mathcal{A}$ be given by the spans of $0 \in \mathbb{R}^4$ and the respective lines in the "Hopf-arrangement", where $\mathbb{R}^3$ is to be considered as the subspace $\mathbb{R}^3 \times \{1\} \subset \mathbb{R}^4$. Figure 2.6 shows the intersection poset of $\mathcal{A}$. The complex $\Delta(P_{<0})$ consists of two circles.
A possible image of these two circles under a model map in the link is shown in Figure 2.5 by the dashed lines. Now the Alexander duals of the two generating cycles in $\tilde{H}_1(\Delta(P_{<0}))$ multiply non-trivially, although $0 + 0 \neq \mathbb{R}^4$.

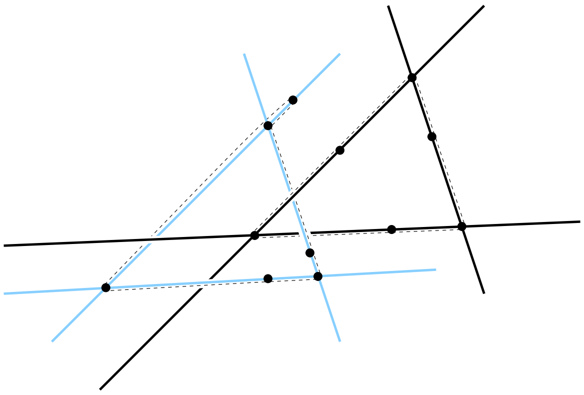


Figure 2.5: The "Hopf-arrangement" in $\mathbb{R}^3$

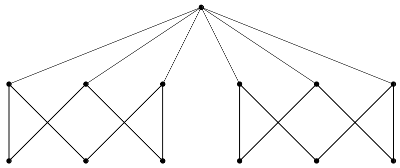


Figure 2.6: The intersection poset of the arrangement $\mathcal{A}$

A more general example

In the last example one might suspect that there might have been a more clever choice of generic points circumvent the non-trivial muliplication. Here we give yet another example in which any choice of generic points gives rise to non-trivial multiplication in the case where the codimension condition is not satisfied. Consider four lines through the origin in $\mathbb{R}^2$ as shown in Figure 2.7.

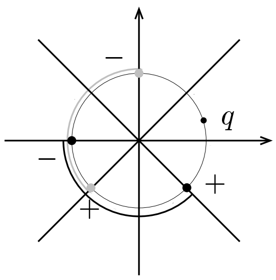


Figure 2.7: Any choice of generic points leads to multiplication even if the codimension condition is not satisfied.

Fix a choice x of generic points, for example the one sketched in the Figure. Now whichever point q you choose for Alexander duality, go from there on, say clockwise, around the 1-sphere. Picking the first and third, respectively the second and fourth generic point, suitably oriented they yield homology classes $\Psi^x(a)$ (given by the black dots with orientations), respectively $\Psi^x(b)$ (given by the grey dots with orientations), where a and b are elements in $\tilde{H}_*(\Delta(P_{<0}))$. But these two classes multiply non-trivially. In fact, the linking product is given by the second and third point suitably oriented.

2.11 General real arrangements are not as bad as you might think

The last section showed that one cannot expect to have a theorem like Theorem 2.9.3 for more than (≥ 2)-arrangements. But even for a general real arrangements, the combinatorics of the arrangement describe some aspects of the linking product structure of its link.

2.11.1 A filtration of the homology of the link

Let $\mathcal{A}$ be a real linear subspace arrangement with intersection poset P. Consider the following filtration of $\tilde{H}_*(L_{\mathcal{A}})$. For $u \in P$ define

$$F_u = F_u(\tilde{H}_*(L_{\mathcal{A}})) = \mathrm{Im}\left(\tilde{H}_*(L_{\mathcal{A}_u}) \xrightarrow{\mathrm{incl}_*} \tilde{H}_*(L_{\mathcal{A}})\right),$$

where $\mathcal{A}_u = \{w \in P : u \subset w\}$.
We want to use the filtration to relate the combinatorially defined ring $(R = \bigoplus_{u\in P} \tilde{H}_*(\Delta(P_{<u})), \circ)$ with its combinatorial linking product and the linking product on the homology of the link.

The topological linking product respects the filtration

Lemma 2.11.1. Let $a \in F_u$ and $b \in F_v$. Then $a \circ b \in F_{u\cap v}$ for the topological linking product.

Proof. $L_{\mathcal{A}_u} \cup L_{\mathcal{A}_v} \subset L_{\mathcal{A}_{u\cap v}}$. □

We note a direct consequence.

Lemma 2.11.2. Let $u, v \in P$ satisfy the codimension condition, i.e., $u + v = \mathbb{R}^n$. Assume $a \in F_u$ and $b \in \sum_{w<v} F_w$, then $a \circ b \in \sum_{w<u\cap v} F_w$. □

2.11.2 The associated graded ring

Associated to the filtration $(F_u)_{u\in P}$ is the graded abelian group given by

$$G_u = F_u \Big/ \sum_{w<u} F_w\,, \qquad u \in P.$$

Define the following product $\diamond$ on $G = \bigoplus_{u \in P} G_u$.

$$\begin{aligned} \diamond \colon G_u \otimes G_v &\to G_{u \cap v} \\ [a] \diamond [b] &:= \begin{cases} [a \circ b], & \text{if } u + v = \mathbb{R}^n, \\ 0, & \text{else.} \end{cases} \end{aligned}$$

Lemma 2.11.2 and Lemma 2.11.1 show that $\diamond$ is well-defined.

2.11.3 Relation to Goresky-MacPherson isomorphisms

We can describe the filtration in terms of a Goresky-MacPherson isomorphism.

Lemma 2.11.3. Let x be any choice of generic points. Then

$$F_u(\tilde{H}_*(L_{\mathcal{A}})) = \bigoplus_{w \leq u} \Psi^x \left(\tilde{H}_*(\Delta(P_{<w})) \right).$$

Proof. Let x be a choice of generic points for the arrangement $\mathcal{A}$. This yields of course a family of generic points $x|_u$ for $\mathcal{A}_u$. The commutativity of the following diagram implies the lemma.

$$\begin{tikzcd} \bigoplus_{w \leq u} \tilde{H}_*(\Delta(P_{<w})) \arrow[r, "\Psi^{x|u}", "\cong"'] \arrow[d, "\mathrm{incl}"] & \tilde{H}_*(L_{\mathcal{A}_u}) \arrow[d, "\mathrm{incl}_*"] \\ \bigoplus_{w \in P} \tilde{H}_*(\Delta(P_{<w})) \arrow[r, "\Psi^x", "\cong"'] & \tilde{H}_*(L_{\mathcal{A}}) \end{tikzcd}$$

$\square$

Note that although the direct sum decomposition depends on the choice of generic points, the filtration does not. The following proposition gives important information on this dependence. It describes the effect of changing generic points on the images of homology classes. It plays the role for general real arrangements that Proposition 2.9.2 plays for (≥ 2)-arrangements.

Proposition 2.11.4. Let x and x' be choices of generic points for the arrangement $\mathcal{A}$, and let $a \in \tilde{H}_*(\Delta(P_{<u}))$. Then

$$\Psi^x(a) - \Psi^{x'}(a) \in \sum_{w<u} F_w.$$

Proof. It is sufficient to consider choices of points x and x' that differ for a single $v \in P$ only. Since the construction of ϕ_u, that determines the restriction of Ψ to $\tilde{H}_*(\Delta(P_{<u}))$, depends only on elements $w \in P$ with $w < u$, we may furthermore assume $v < u$. Now consider the subcomplex C of $\Delta(P_{<u})$ consisting of all simplices containing v. Every vertex w of C is comparable with

v, that is, either contains v or is contained by v. Therefore $x_w \in L_{\mathcal{A}_v}$ for every vertex w of C, and of course $x'_v \in L_{\mathcal{A}_v}$. This means that the restrictions of $\iota_u * \phi_u$ as well as of $\iota_u * \phi'_u$ to $\mathbb{S}^{\dim u-1} * C$ have their image in $L_{\mathcal{A}_v}$. Therefore $\Psi^x(a) - \Psi^{x'}(a) = \big((\iota_u * \phi_u)_* - (\iota_u * \phi'_u)_*\big)([\mathbb{S}^{\dim u-1}] * a) \in F_v$. □

Now we only have to put things together to get the main result of this section.

Theorem 2.11.5. *Any Goresky-MacPherson isomorphism Ψ^x for $\mathcal{A}$ induces a ring isomorphism*

$$\bar{\Psi}\colon R \to G$$

by

$$\begin{aligned} \tilde{H}_*(\Delta(P_{<u})) &\to G_u \\ a &\mapsto [\Psi^x(a)], \end{aligned}$$

and any two of them are equal.

Proof. The fact that $\Psi^x\colon R \to \tilde{H}_*(L_{\mathcal{A}})$ is an isomorphism of abelian groups together with Lemma 2.11.3 implies that the induced homomorphism $\bar{\Psi}\colon R \to G$ is also an isomorphism of abelian groups. Because of Proposition 2.11.4 it is independent of x. Because of this independence, Proposition 2.7.4 suffices to ensure that $\bar{\Psi}$ is a ring homomorphism. □

CHAPTER 3

THE NEIGHBORHOOD COMPLEXES OF STABLE KNESER GRAPHS

This chapter is joint work with Anders Björner [BL99].

3.1 Introduction

In 1955 Martin Kneser [Kne55] conjectured that if one splits the n-subsets of a $(2n+k)$-element set into $k+1$ classes, then one of the classes contains two disjoint n-subsets. In 1978 László Lovász proved this conjecture [Kne55] – in graph language a question about the chromatic number of the Kneser graphs – by introducing the concept of a neigborhood complex of a graph (cf. [Lov78]). He applied the Borsuk-Ulam theorem (cf. [Bre93, p. 240*ff*]) to show that if the neigborhood complex of a graph is topologically $(k-1)$-connected, then the graph is not $(k+1)$-colorable. Furthermore, he showed that the neighborhood complex of the Kneser graph $KG_{n,k}$ is $(k-1)$-connected, and thus proved the Kneser conjecture.

Shortly afterwards Imre Bárány [Bár78] provided a very elegant and short proof also applying the Borsuk-Ulam theorem. In the same year Alexander Schrijver used Bárány's method to obtain a family of vertex critical subgraphs (cf. [Sch78]) – the family of stable Kneser graphs $SG_{n,k}$.

It is a natural question if the chromatic number of these subgraphs $SG_{n,k}$ can be obtained by Lovász method. In fact, one could expect a simple structure for these complexes. They would have to be $(k-1)$-connected and not more. The most natural example for such a space is a sphere of dimension k. It turned out that this is always the case.

An interesting property of the neighborhood complexes of the family of stable Kneser graphs $SG_{2,k}$ is that they contain the simplicial complex encoding triangulations of a $(k+4)$-gon – as a complex the boundary of the appropriate associahedron – as a deformation retract. We will show this in the last section.

3.2 Preliminaries

We recall the definition of a Kneser graph, its vertex critical subgraphs defined by Schrijver, and the notion of a neighborhood complex of a graph.
Let $n \geq 1$ and $k \geq 0$.

- ▷ The vertices of the *Kneser graph* $KG_{n,k}$ are given by the n-subsets of $[2n+k] = \{1, \ldots, 2n+k\}$; two of them are joined by an edge iff they are disjoint, see Figure 3.1. In 1955 Kneser asked if the chromatic number of $KG_{n,k}$ is $k+2$ (cf. [Kne55]).

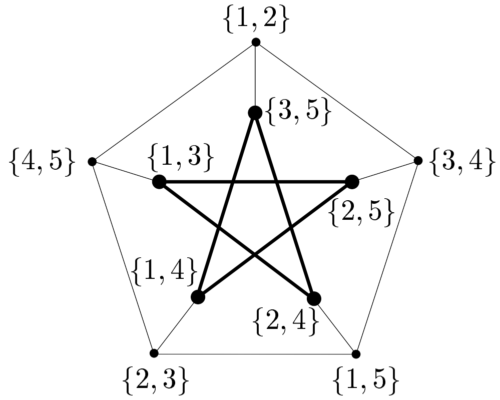


Figure 3.1: The Kneser graph $KG_{2,1}$ and the stable Kneser graph $SG_{2,1}$.

- ▷ The *neighborhood complex of a graph* $G = (V, E)$ is a simplicial complex on the vertex set V and its simplices are given by sets of vertices that have a common neighbor.

- ▷ A subset $v \subset [2n+k]$ is *quasistable* if for all $i \in [2n+k-1]$ the set $\{i, i+1\}$ is not contained in v. The set v is *stable* if it is quasistable and it does not contain the set $\{1, 2n+k\}$, i.e., a subset is stable if it does not contain two neighbors in the cyclic ordering of $[2n+k]$.

- ▷ The vertices of the *stable Kneser graph* $SG_{n,k}$ (introduced in 1978 by A. Schrijver) are the stable n-subsets of $[2n+k]$; two of them are joined by an edge iff they are disjoint. $SG_{n,k}$ is an induced subgraph of $KG_{n,k}$, and with respect to the chromatic number it is vertex critical [Sch78]. In Figure 3.1 the stable Kneser graph $KG_{2,1}$ is indicated by the bold vertices and edges.

- ▷ The neighborhood complex of $SG_{n,k}$ is hence given by

$$\Sigma_{n,k} = \{\{v_1, \ldots, v_s\} : \forall i (v_i \text{ vertex of } SG_{n,k}) \exists \text{ vertex } v (\forall i (v \cap v_i = \emptyset))\},$$

i.e., the faces of $\Sigma_{n,k}$ are given by any family of stable n-sets in the complement of a stable n-set.

▷ For each stable n-set $v \subset [2n+k]$ – a vertex of $SG_{n,k}$ – we define the *neighbor facet* $\Delta_v^{n,k}$ *of* v:

$$\Delta_v^{n,k} = \{w : w \subset [2n+k] \text{ stable } n\text{-set}, v \cap w = \emptyset\}.$$

We will omit the superscript whenever it will not cause confusion. The neighbor facets constitute the facets of $\Sigma_{n,k}$. We thus obtain the following description of $\Sigma_{n,k}$:

$$\Sigma_{n,k} = \{F \subset \Delta_v : v \subset [2n+k] \text{ stable } n\text{-set}\}.$$

Examples

▷ The only stable n-sets of $[2n+0]$ are $\{1,3,5,\dots,2n-1\}$ and $\{2,4,6,\dots,2n\}$ and they are complementary. Hence $\Sigma_{n,0}$ is homeomorphic to the 0-sphere $\mathbb{S}^0$.

▷ Each stable n-set of $[2n+1]$ is characterized by a pair $\{i, i+1\} \pmod{2n+1}$ of free positions in its complement. Each such admits two stable n-sets in its complement constituting a 1-simplex of $\Sigma_{n,1}$. Together these 1-simplices yield a 1-sphere, and thus $\Sigma_{n,1}$ is homeomorphic to $\mathbb{S}^1$, cf. Figure 3.1.

▷ Stable 1-sets are just 1-sets by definition. Hence for $\Sigma_{1,k}$ we obtain

$$\{\{v_1,\dots,v_s\} : \forall i(v_i \subset [k+2] \text{ 1-set } \exists \text{ 1-set } v \subset [k+2](\forall i(v \cap v_i = \emptyset))\}.$$

This complex is the boundary of a $(k+1)$-dimensional simplex. Therefore $\Sigma_{1,k}$ is homeomorphic to the k-sphere $\mathbb{S}^k$.

In general we can not expect $\Sigma_{n,k}$ to be homeomorphic to the k-sphere: for example $\Sigma_{2,2}$ is a pure 3-dimensional complex. Figure 3.2 illustrates the two prototypes $\Delta_{\{1,3\}}$ and $\Delta_{\{1,4\}}$ of facets of $\Sigma_{2,2}$.

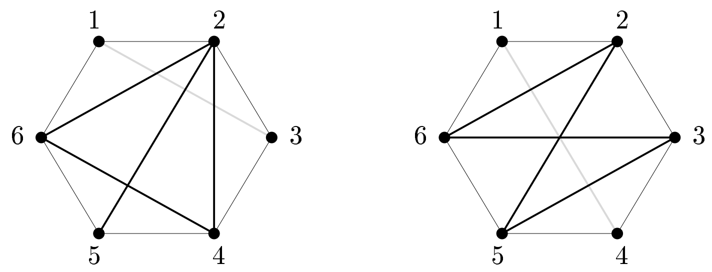


Figure 3.2: The black diagonal edges in each polygon yield the vertices of a 3-simplex in $\Sigma_{2,2}$.

But in general the complexes $\Sigma_{n,k}$ are not even pure as can be seen for example by considering the two prototypes of facets of $\Sigma_{3,2}$.

3.3 The neighborhood complexes of stable Kneser graphs are homotopy spheres

Theorem 3.3.1. *The simplicial complex $\Sigma_{n,k}$ is homotopy equivalent to the k-sphere $\mathbb{S}^k$ for all $n \geq 1$ and $k \geq 0$.*

The proof of the Theorem proceeds by induction on k. We will cover the complex $\Sigma_{n,k}$ by two contractible subcomplexes that intersect up to homotopy in $\Sigma_{n,k-1}$. Let for all $n, k \geq 1$ the subcomplexes $A_{n,k}$ and $B_{n,k}$ of $\Sigma_{n,k}$ be defined by:

$$A_{n,k} = \{F \subset \Delta_v : v \text{ vertex of } \Sigma_{n,k} \text{ such that } 1 \notin v\}$$
$$B_{n,k} = \{F \subset \Delta_v : v \text{ vertex of } \Sigma_{n,k} \text{ such that } 1 \in v\}.$$

Obviously the union of $A_{n,k}$ and $B_{n,k}$ is $\Sigma_{n,k}$.

Proposition 3.3.2. There are the following homotopy equivalences.

(i) $A_{n,k} \simeq *$, for all $n, k \geq 1$,

(ii) $B_{1,k} \simeq *$, for all $k \geq 1$,

(iii) $B_{n,k} \simeq B_{n-1,k}$ for all $n \geq 2$, and

(iv) $A_{n,k+1} \cap B_{n,k+1} \simeq \Sigma_{n,k}$ for all $n \geq 1$, $k \geq 0$.

I.e., $A_{n,k}$ and $B_{n,k}$ are contractible to a point, and they intersect up to homotopy in a neighborhood complex of a stable Kneser graph of dimension one less.

The Theorem can be deduced by induction from the Proposition using the Corollary of the following Lemma. The generality in which we state it here is not needed for the Corollary but will be needed in the next section. If a space X is covered by a family $(X_\alpha)_{\alpha\in A}$ we use the notation $X_\sigma = \bigcap_{\alpha\in\sigma} X_\alpha$ for subsets $\sigma \subset A$.

Gluing Theorem ([tD71]). Let $f{:}X \to Y$ be a continuous map, $(X_\alpha)_{\alpha\in A}$ and $(Y_\alpha)_{\alpha\in A}$ be closed, finite-dimensional, locally finite coverings of X and Y, respectively. Assume that the inclusions $X_\sigma \hookrightarrow X_\tau$ respectively $Y_\sigma \hookrightarrow Y_\tau$ are cofibrations for finite subsets $\sigma \subset \tau \subset A$. If $f(X_\alpha) \subset Y_\alpha$ and $f|_{X_\sigma}{:}X_\sigma \to Y_\sigma$ is a homotopy equivalence, then f itself is a homotopy equivalence.

Corollary. Let X be a simplicial (or CW-) complex and A and B contractible subcomplexes with $A \cup B = X$. Then X is homotopy equivalent to the suspension $\Sigma(A \cap B)$.

Proof of the Proposition. (i) For notational reasons we prove that

$$A'_{n,k} = \{F \subset \Delta_v : v \text{ vertex of } \Sigma_{n,k} \text{ such that } 2n+k \notin v\}$$

is contractible. In fact we show that it is collapsible using the following generalization of the concept of a cone (cf. [BW99]). It can also be viewed as a complete Morse matching (cf. [For99]).

Lemma 3.3.3. Let $\Gamma_1 \subset \cdots \subset \Gamma_l = \Gamma$ be simplicial complexes, and let $\Gamma_0 = \emptyset$. Assume there exist vertices $w_1, \ldots, w_l$ such that for $i = 1, \ldots, l$ the assignment

$$F \longmapsto \begin{cases} F \cup \{w_i\} & \text{, if } w_i \notin F, \\ F \setminus \{w_i\} & \text{, if } w_i \in F. \end{cases}$$

maps $\Gamma_i \setminus \Gamma_{i-1}$ into itself. Then Γ is collapsible.

In order to use this Lemma we begin by defining a sequence $\Gamma_1 \subset \cdots \subset \Gamma_l = A'_{n,k}$ of ascending subcomplexes of $A'_{n,k}$. To do so order all stable n-sets $v \subset [2n+k]$, $2n+k \notin v$ lexicographically. Say $v_1 \prec \cdots \prec v_l$. For $i = 1, \ldots, l$ define

$$\Gamma_i = \{F \subset \Delta_{v_j} : 1 \leq j \leq i\},$$

and let $\Gamma_0 = \emptyset$.
Next we define a set $w_1, \ldots, w_l$ of vertices. Consider $v_i = \{a_1, \ldots, a_n\} \subset [2n+k-1]$, and define the stable n-set $w_i = \{a_1+1, \ldots, a_n+1\} \subset [2n+k]$, $i = 1, \ldots, l$. Note that $w_i \in \Delta_{v_i}$ for $i = 1, \ldots, l$. See Figure 3.3.

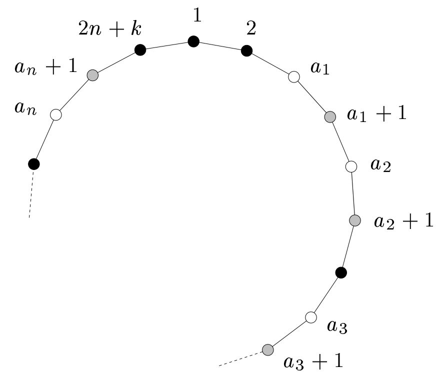


Figure 3.3: The vertices v_i and w_i.

The last step is to consider the map described in the Lemma. Let $i \in \{1, \ldots, l\}$ and $F \in \Gamma_i \setminus \Gamma_{i-1}$ a simplex. If $w_i \notin F$ we map F to $F \cup \{w_i\}$. It is easy to see that $F \cup \{w_i\} \in \Gamma_i \setminus \Gamma_{i-1}$. If $w_i \in F$ then we map F to $F \setminus \{w_i\}$. In this case $F \setminus \{w_i\} \in \Gamma_i \setminus \Gamma_{i-1}$ for the following reason. Consider the *support*

$\operatorname{supp}(F) = \bigcup F \subset [2n+k]$ of F. The fact that $F \in \Gamma_i \setminus \Gamma_{i-1}$ implies that the lexicographic smallest stable n-set in $[2n+k] \setminus \operatorname{supp}(F)$ is v_i. Furthermore, $w_i \in F$ implies that the first n elements of $[2n+k] \setminus \operatorname{supp}(F)$ are given by the set v_i. Hence $F \setminus \{w_i\} \in \Gamma_{i-1}$ only if the set $\{a_1, a_1+1, a_2, a_2+1, \dots, a_n, a_n+1\}$ contains a stable n-set that precedes v_i in the lexicographic order. But this is not the case.
(ii) The complex

$$\begin{aligned} B_{1,k} &= \{F \subset \Delta_v^{1,k} : v \subset [2+k] \text{ 1-set}, 1 \in v\} \\ &= \{F \subset \Delta_{\{1\}}^{1,k}\} \end{aligned}$$

is a k-dimensional simplex and therefore contractible.
(iii) We want to apply the

Nerve Theorem (see for example [Bjö94b] or [McC67]). Let Γ be a simplicial complex and $(\Gamma_i)_{i \in I}$ a family of subcomplexes such that $\Gamma = \bigcup_{i \in I} \Gamma_i$ and every finite intersection $\Gamma_{i_1} \cap \dots \cap \Gamma_{i_s}$ is contractible. Then the *nerve complex*

$$\mathcal{N}(\Gamma_i) := \left\{ \sigma \subset I : \sigma \text{ finite}, \bigcap_{i \in \sigma} \Gamma_i \neq \emptyset \right\}$$

is homotopy equivalent to Γ. $\square$

Consider the covering $(2^{\Delta_w})_{w \in \{v \subset [2n+k] : v \text{ stable } n\text{-set}, 1 \in v\}}$ of $B_{n,k}$, where 2^{Δ_w} is an abbreviation for the complex $\{F : F \subset \Delta_w\}$ of all faces of Δ_w. By the Nerve Theorem we obtain

$$\begin{aligned} B_{n,k} \simeq \mathcal{N}(2^{\Delta_w}) &= \big\{\{v_1, \dots, v_s\} : v_i \text{ stable}, 1 \in v_i, 2^{\Delta_{v_1}} \cap \dots \cap 2^{\Delta_{v_s}} \neq \emptyset\big\} \\ &= \{\{v_1, \dots, v_s\} : v_i \text{ stable}, 1 \in v_i, \exists \text{ stable } v (\forall i (v \cap v_i = \emptyset))\} \\ &= \{\{v_1, \dots, v_s\} : v_i \text{ stable}, 1 \in v_i, \\ &\qquad \exists \text{ stable } v (2 \in v, 2n+k \in v \, \forall i (v \cap v_i = \emptyset))\} \end{aligned}$$

where the last equation follows by the stability of the vertices. Hence by deleting the element $1 \in [2n+k]$ and identifying $2, 2n+k \in [2n+k]$ (see Figure 3.4) we obtain the following identification:

$$\begin{aligned} \mathcal{N}(2^{\Delta_w}) &\cong \big\{F \subset \Delta_v^{n-1,k} : v \subset [2(n-1)+k] \text{ stable } (n-1)\text{-set}, 1 \in v\big\} \\ &= B_{n-1,k}. \end{aligned}$$

(iv) In order to get a good description of the intersection $A_{n,k+1} \cap B_{n,k+1}$ we need the following

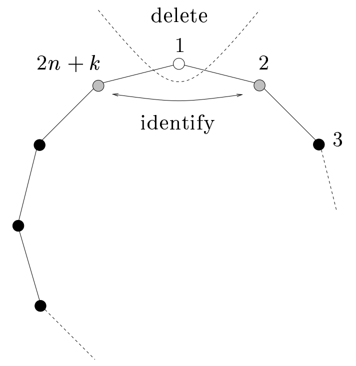


Figure 3.4: Deletion and identification of elements in $[2n+k]$.

Lemma 3.3.4. Let $v_w, v_b \subset [2n+k+1]$ be two stable n-subsets such that $1 \notin v_w$ and $1 \in v_b$. Then there exists a stable n-subset $v \in [2n+k+1]$ with the following properties:

(i) $v \subset v_w \cup v_b$,

(ii) $1 \notin v$, and

(iii) $2 \notin v$ or $2n+k+1 \notin v$.

Proof. Call $i \in [2n+k+1]$ black if $i \in v_b$ and white if $i \in v_w$. In general, it can happen that i is black and white.
Case (1): $2 \notin v_w$ or $2n+k+1 \notin v_w$. Set $v = v_w$.
Case (2): $2 \in v_w$ and $2n+k+1 \in v_w$. Consider the sequence $2, 3, 4, \ldots$. By the stability of v_b and v_w the numbers in the sequence are colored white and black alternately until there is a non-colored number. Non-colored numbers exist since $2n+k+1 > 2n$. Let i be the smallest number such that $i+1$ is not colored (see Figure 3.5).

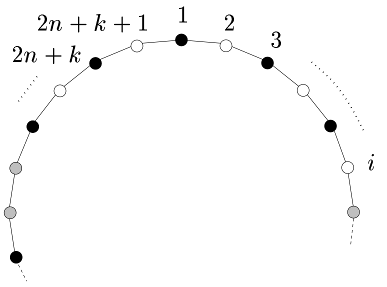


Figure 3.5: Points in $[2n+k+1]$ colored alternately.

If i is white, then set $v = \{2, 4, \ldots, i\} \cup \{\text{all black numbers } > i\}$, and if i is black, then set $v = \{3, 5, \ldots, i\} \cup \{\text{all white numbers } > i\}$. $\square$

We compute the intersection

$$\begin{aligned}
&A_{n,k+1} \cap B_{n,k+1} = \\
&= \left\{F \subset \Delta_{v_w}^{n,k+1} \cap \Delta_{v_b}^{n,k+1} : v_w, v_b \subset [2n+k+1] \text{ stable } n\text{-sets, } 1 \notin v_w,\ 1 \in v_b\right\} \\
&= \{\{v_1, \ldots, v_s\} : \forall i (v_i \subset [2n+k+1] \text{ stable } n\text{-set, } 1 \notin v_i),\ \exists \text{ stable } n\text{-set } v \\
&\qquad \text{such that } 1 \notin v \text{ and } (2 \notin v \text{ or } 2n+k+1 \notin v), \forall i (v \cap v_i = \emptyset)\},
\end{aligned}$$

where the last equation is justified by the Lemma. Now we delete the number $1 \in [2n+k+1]$, since it is not used for the vertices in the intersection. This forces us to consider quasistable n-sets as vertices.

$$\begin{aligned}
A_{n,k+1} \cap B_{n,k+1} \cong \{\{v_1, \ldots, v_s\} &: \forall i (v_i \subset [2n+k] \text{ quasistable } n\text{-set}), \\
&\exists \text{ stable } n\text{-set } v \subset [2n+k], \forall i (v \cap v_i = \emptyset)\} \\
= \{F \subset \bar{\Delta}_v^{n,k} &: v \subset [2n+k] \text{ stable } n\text{-set}\},
\end{aligned}$$

where $\bar{\Delta}_v^{n,k} := \{w : w \subset [2n+k] \text{ quasistable } n\text{-set}, v \cap w = \emptyset\}$. Denote by $I_{n,k}$ this identified intersection of $A_{n,k+1} \cap B_{n,k+1}$.

Observation. ▷ $\Sigma_{n,k} \subset I_{n,k}$ and hence a copy of $\Sigma_{n,k}$ is contained in $A_{n,k+1} \cap B_{n,k+1}$.

▷ $\{v : v \subset [2n+k] \text{ quasistable } n\text{-set, } 1, 2n+k \in v\}$ are the vertices of $I_{n,k}$ not used by $\Sigma_{n,k}$.

In order to describe $I_{n,k}$ in terms of $\Sigma_{n,k}$ we define two subcomplexes $C_{n,k}$ and $D_{n,k}$ which measure the surplus.

$$\begin{aligned}
C_{n,k} &= \left\{F \subset \bar{\Delta}_v^{n,k} : v \subset [2n+k] \text{ stable } n\text{-set, } 1, 2n+k \notin v\right\} \\
D_{n,k} &= \left\{F \subset \Delta_v^{n,k} : v \subset [2n+k] \text{ stable } n\text{-set, } 1, 2n+k \notin v\right\}.
\end{aligned}$$

The facets of $C_{n,k}$ constitute all facets of $I_{n,k}$ containing vertices of $I_{n,k}$ not in $\Sigma_{n,k}$. Hence we have

$$I_{n,k} = \Sigma_{n,k} \cup C_{n,k}.$$

The intersection of $\Sigma_{n,k}$ and $C_{n,k}$ is given by simplices of $\Sigma_{n,k}$ that are contained in a facet of $C_{n,k}$ and therefore

$$D_{n,k} = \Sigma_{n,k} \cap C_{n,k}.$$

In order to show the homotopy equivalence

$$I_{n,k} \simeq \Sigma_{n,k}$$

it suffices to prove that $C_{n,k}$ and $D_{n,k}$ are contractible. The sufficiency can be seen for example by using the following

Contraction Lemma (see [Bre93, Chapter VII.]). If $f{:}A \longrightarrow X$ is a cofibration and A is contractible, then the collapse $X \longrightarrow X/A$ is a homotopy equivalence.

The contractibility of $C_{n,k}$ and $D_{n,k}$ is shown by using the analogous generalized cone construction argument that we used for the contractibility of $A_{n,k}$. □

3.4 The neighborhood complexes and associahedra

In the case $n = 2$ the stable n-subsets of $[k+4]$, i.e., the vertices of $\Sigma_{2,k}$, correspond to diagonal edges of a $(k{+}4)$-gon. For any stable 2-set $v \subset [k{+}4]$ the simplex Δ_v contains faces that correspond to triangulations of the $(k+4)$-gon, compare Figure 3.6. In fact, the simplicial complex Θ_k constituted by all k-dimensional simplices in $\Sigma_{2,k}$ that correspond to triangulations is a triangulated sphere. It was shown by Haiman [Hai84] and Lee [Lee89] that this sphere arises as the boundary complex of a $(k+1)$-dimensional simplicial polytope, which is called *associahedron* for the fact that triangulations of the $(k+4)$-gon correspond to ways of parenthesizing a sequence of $k+3$ symbols. We show that the subcomplex Θ_k of $\Sigma_{2,k}$ is a strong deformation retract.

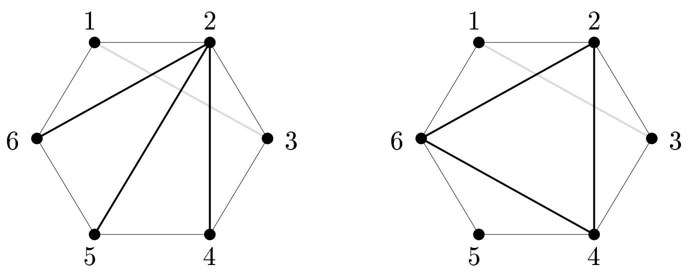


Figure 3.6: The triangulations in the facet $\Delta_{\{1,3\}}$.

Coverings of $\Sigma_{2,k}$ and Θ_k

Consider the covering $(2^{\Delta_v})_{v \text{ stable}}$ of $\Sigma_{2,k}$ and the induced covering $(T_v)_{v \text{ stable}}$ of Θ_k, where $T_v = \Theta_k \cap 2^{\Delta_v}$. For example in the case $k = 2$ the faces of $\Delta_{\{1,3\}}$ given by triangulations shown in Figure 3.6 yield the facets of $T_{\{1,3\}}$.

Lemma 3.4.1. For all $\sigma \subset \{v : v \text{ stable 2-set of } [k+4]\}$ the following inclusion is a homotopy equivalence

$$i : \bigcap_{v\in\sigma} T_v \hookrightarrow \bigcap_{v\in\sigma} 2^{\Delta_v}.$$

Proof. It suffices to show that for all σ

- ▷ the space $\bigcap_{v\in\sigma} T_v$ is empty if and only if $\bigcap_{v\in\sigma} 2^{\Delta_v}$ is empty,
- ▷ and $\bigcap_{v\in\sigma} T_v$ is contractible in the case where it is non-empty.

The first statement is clear. The second statement follows from the fact that any space $\bigcap_{v\in\sigma} T_v$ is a cone as can be seen as follows. Consider a maximal sequence of consecutive numbers in $\bigcup_{v\in\sigma} v \subset [k+4]$ modulo $k+4$. The edge given by the predecessor and successor modulo $k+4$ of this sequence is contained in every facet of $\bigcap_{v\in\sigma} T_v$ (cf. Figure 3.7). □

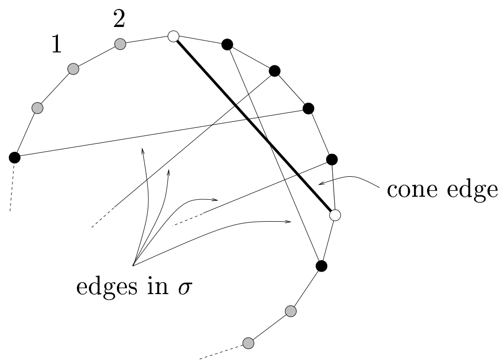


Figure 3.7: The cone peak edge.

The Gluing Theorem (page 66) tells us now that the inclusion map $i{:}\Theta_k \hookrightarrow \Sigma_{2,k}$ is a homotopy equivalence, i.e., Θ_k is a weak deformation retract of $\Sigma_{2,k}$. Since $(\Sigma_{2,k}, \Theta_k)$ is a pair of simplicial complexes some elementary results from homotopy theory (cf., e.g., [Spa66, p. 31 & p. 402]) imply the following.

Theorem 3.4.2. *The subcomplex Θ_k constituted by all simplices in $\Sigma_{2,k}$ that correspond to triangulations of the $(k+4)$-gon is a strong deformation retract of $\Sigma_{2,k}$.* □

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