Graph theory 5707
Fall Semester 2001
First Midterm
October 17, 2001, 3:35-4:50

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Problem 1 (10 points): Isomorphisms and complements Let G and H be simple graphs. Show that $G \cong H$ if and only if $\bar{G} \cong \bar{H}$.

Let's denote the vertex sets by $V = V(G) = V(\bar{G})$ and $V' = V(H) = V(\bar{H})$. Now $G \cong H$ if and only if there exists a bijective map $\varphi : V \to V'$ such that

$$uv \in E(G) \iff \varphi(u)\varphi(v) \in E(H),$$

which is equivalent to

$$uv \notin E(G) \iff \varphi(u)\varphi(v) \notin E(H),$$

which in turn is equivalent to

$$uv \in E(\bar{G}) \iff \varphi(u)\varphi(v) \in E(\bar{H}).$$

But $\bar{G}$ is isomorphic to $\bar{H}$ if and only if there exists a bijective map $\varphi : V \to V'$ satisfying this last condition.

Problem 2 (10 points): Shrunk matrices Let G be a graph on n vertices. For $v \in V(G)$ and $e \in E(G)$, describe the adjacency matrices of $G - v$ and $G - e$ in terms of the adjacency matrix of G.

If the rows and columns of the adjacency matrix $A = (a_{ij})$ of G are labeled by the vertices $v_1, \dots, v_n$, and v corresponds to v_i then the adjacency matrix of $G - v$ is given by A with the row and column labeled by v_i deleted. If furthermore the endpoints of e are the vertices v_k and v_l then the adjacency matrix of $G - e$ is given by the $n \times n$-matrix $B = (b_{ij})$ with

$$b_{ij} = \begin{cases} a_{ij} - 1, & \text{if } \{i, j\} = \{k, l\}, \\ a_{ij}, & \text{otherwise.} \end{cases}$$

Problem 3 (10 points): Euler's edges Prove or disprove: Every Eulerian bipartite graph has an even number of edges.

Let X, Y be the bipartition of the graph. Then the number of edges is $\sum_{v \in X} d(v)$, which is even since $d(v)$ is even for every vertex v for the graph being Eulerian.

Problem 4 (10 points): Odd vertices Prove or disprove: If u and v are the only vertices of odd degree in a graph G, then G contains a u, v-path.

The degree sum formula implies that the sum of the degrees of the vertices of a graph is even. This certainly holds for each connected component. Hence, the two odd degree vertices have to be in the same connected component, and are therefore connected by a path.

Problem 5 (10 points): In cycles Let G be a digraph with no isolated vertices, and in which indegree equals outdegree at each vertex. Prove that G decomposes into (directed) cycles.

Let's do induction on the number of edges. If G has one edge, then the graph consists of one directed loop. Now let's suppose that G has more than one edge. If G has a loop, remove it. Otherwise, consider a directed path of maximal length. At the end vertex of the path there must be another edge leaving the vertex, since indegree equals outdegree. This edge connects somewhere to the path, creating a directed cycle. Remove this cycle. By induction the remaining graph decomposes into cycles, as we have not changed the property that indegree equals outdegree at each vertex.

Problem 6 (10 points): Different diameters Let G be a simple graph with diameter at least 4. Show that $\bar{G}$ has diameter at most 2.

Since $diam(G) \geq 4$ there is a pair of vertices u, v such that $d_G(u,v) \geq 4$. Let $U = \{w \in V(G) : d_G(u,w) \leq 1\}$ and $V = \{w \in V(G) : d_G(v,w) \leq 1\}$. We want to show that $d_{\bar{G}}(x,y) \leq 2$ for any pair of vertices $x, y \in V(G) = V(\bar{G})$.
Case 1: x, y are both in the complement of U or both in in the complement of V. By symmetry it suffices only to consider $x, y \in V(G) \setminus V$ then xv and vy are edges in $\bar{G}$ and x, v, y gives a path of length 2.
Case 2: One vertex in U or in V and one vertex outside of $U \cup V$. Again, by symmetry we can assume that $x \in U$ and $y \in V(G) \setminus (U \cup V)$. Then xv and vy are edges in $\bar{G}$ and x, v, y gives a path of length 2.
Case 3: One vertex in U and the other in V. By symmetry we can assume $x \in U$ and $y \in V$. Since $d_G(u,v) \geq 4$ any vertex in U has distance at least two from any vertex in V in the graph G, and hence xy is an edge in $\bar{G}$.

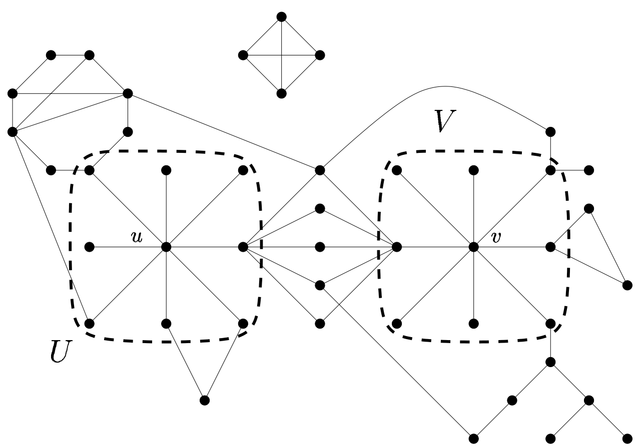

FIGURE 1. The sets U and V in G.

Problem 7 (20 points): Simply even Prove that the number of simple even graphs with vertex set $\{1,\ldots,n\}$ is $2^{\binom{n-1}{2}}$.
Remember: a graph is called *even*, if every vertex has even degree, and $\binom{n-1}{2}$ is the number of 2-subsets of an $(n-1)$-set.
Hint: Establish a bijection to the set of all simple graphs with vertex set $\{1,\ldots,n-1\}$.

To construct a simple graph on the vertex set $\{1,\ldots,n-1\}$ we have a choice of chosing any of the possible $\binom{n-1}{2}$ pairs of vertices as an edge or not. Hence there are $2^{\binom{n-1}{2}}$ simple graphs on the vertex set $\{1,\ldots,n-1\}$. Let G be a simple graph on this vertex set. We assign a new simple graph G' on the vertex set $\{1,\ldots,n\}$ to it as follows. Introduce the new vertex n and connect it to all vertices of G of odd degree. Then we have the following:

(1) The graph G' is even since G has an even number of odd degree vertices, that were all connected to n. More detailed, the degree of each odd vertex among $\{1,\ldots,n-1\}$ was increased by one making it an even vertex. And n being connected to an even number of vertices turns out to be an even vertex as well.
(2) The assignment $G \mapsto G'$ is a bijection from the set of all simple graphs on the vertex set $\{1,\ldots,n-1\}$ to the set of all simple even graphs on the vertex set $\{1,\ldots,n\}$. The inverse map is given by deleting the vertex n.

Problem 8 (20 points): Heavy edges
Let C be a cycle in a connected weighted graph. Let e be an edge of maximum weight on C.
a) Prove that there is a minimum spanning tree not containing e.
b) Use this to prove that you obtain a minimum-weight spanning tree if you iteratively delete a heaviest non-cut edge until the remaining graph is acyclic.

a) Consider a minimum spanning tree T. If it does not contain e we are done. If it does contain e, consider $T - e$ which must be disconnected.
This yields a partition of the vertex set of G into two sets. Since e was contained in the cycle C, there must be another edge e' of C with endpoints in both of these sets. Therefore $T - e + e'$ is another spanning tree. Since e' is an edge of the cycle C it has weight at most the weight of e, and hence $T - e + e'$ is a spanning tree of weight at most the weight of T, and hence must be minimal.

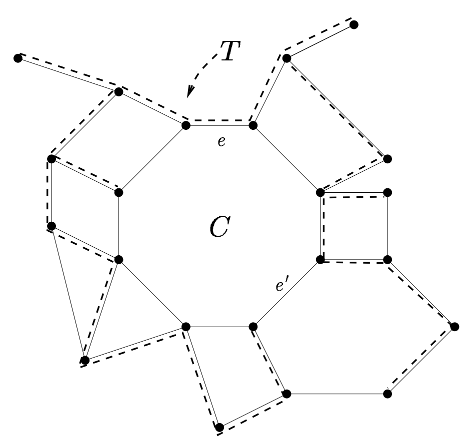

FIGURE 2. The spanning tree T with a heaviest edge e.

b) As the removal of non-cut edges does not increase the number of connected components the remaining graph certainly is a spanning tree. Let's show that the resulting graph is minimal by induction on the number k of non-cut edges. If $k = 0$ the graph itself is a tree, and hence is the minimum spanning tree. If $k > 0$ consider the first edge e chosen by the algorithm. We know that there is a minimum spanning tree not containing e. Hence remove e from the graph thereby reducing the number of non-cut edges. By induction we know that the algorithm now produces a minimum spanning tree.