Graph theory 5707
Fall Semester 2001
Second Midterm
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Problem 1 (20 points): Saturation Let S be the set of vertices saturated by a matching M in a graph G. Prove that some *maximum matching* also saturates all of S. Is this true for every maximum matching?

We show that if M is not a maximum matching, then we can construct a matching with one more edge still saturating S. If M is not a maximum matching, then it has an M-augmenting path P. Now exchange the edges of P in M with the edges of P that are not in M to obtain a matching M' of larger size. Note that all vertices that were saturated by M are still saturated by M'. Inductively we obtain a maximum matching saturating a set of vertices containing S.
In general, not every maximum matching must saturate the set S, as can be seen by considering a path with an odd number of vertices and M being the first edge of the path.

Problem 2 (15 points): Independence Recall that we defined $\alpha(G)$ to be the maximum size of an independent set of vertices in G, and $\Delta(G)$ to be the maximal degree among the vertices of G. Prove that

$$\alpha(G) \geq \frac{n(G)}{\Delta(G)+1}$$

for every loopless graph G on $n(G)$ vertices.

If I is an independent set of vertices of maximum size $\alpha(G)$, then

$$\bigcup \{\{v\} \cup N(v) : v \in I\} = V(G)$$

by maximality of I. Hence

$$\alpha(G)(\Delta(G)+1) \geq |\bigcup \{\{v\} \cup N(v) : v \in I\}\,| = |V(G)| = n(G).$$

Problem 3 (15 points): King in the hall Let G be a bipartite X,Y-graph. Use the König–Egerváry Theorem to prove Hall's Theorem for G.

Let's assume Hall's condition, i.e., $|N(S)| \geq |S|$ for all $S \subseteq X$, holds. It suffices to find a matching of size $|X|$. For that we use the König–Egerváry Theorem and show that the minimal size of a vertex cover is at least $|X|$. Assume that G has a minimal vertex cover C with fewer than $|X|$ elements. Define $A = X \cap C$ and $B = Y \cap C$, and note that $|A| + |B| = |C| < |X|$. Since C is a vertex cover there can be no edge from $X \setminus A$ to $X \setminus B$ and hence

$$|N(X \setminus A)| \leq |B| < |X| - |A| = |X \setminus A|.$$

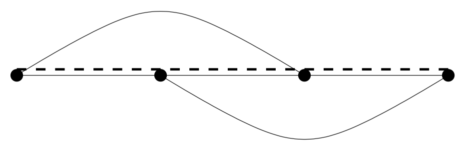
A contradiction.

Problem 4 (15 points): Too connected Let G be a connected graph with at least three vertices. Form G' from G by adding an edge with endpoints x, y whenever $d_G(x, y) = 2$ for a pair of vertices $x, y \in V(G)$. Prove that G' is 2-connected.

Since G' has at least three vertices G' with one vertex deleted has at least two vertices. Now assume that $G' \setminus v$ is disconnected for some $v \in V(G)$. Then in particular $G \setminus v$ is disconnected. There are vertices a and b in different connected components of $G' \setminus v$. Consider a path P in the connected graph G from a to b. It must pass through the vertex v. Now the vertices along the path right before and right after v must be connected by an edge in G'. Hence we can shorten P and obtain an a, b-path in $G' \setminus v$. Contradiction.

Problem 5 (15 points): Internally disjoint? Prove or disprove: If P is a u, v-path in a 2-connected graph G, then there is a u, v-path Q internally disjoint from P.

Consider $G = P_4$ the path on four vertices. If we construct G' as in Problem 4 we obtain the graph shown below, which then is 2-connected. If we consider the path P indicated by the dashed line, then there is no other path with the same endpoints internally disjoint to P.



Problem 6 (20 points): Fan lemma Let G be a k-connected graph. Use Menger's theorem to prove that for any choice of $U \subseteq V(G)$, $|U| \geq k$, and $v \in V(G) \setminus U$ there are k many v-U-paths such that any two of them only share the vertex v.

Consider the two sets U and $N(v)$. Since G is k-connected $|N(v)| \geq k$, and the two sets can not be separated with less than k vertices, otherwise there would be a vertex in U and a vertex in $N(v)$ that could be separated by less than k vertices. Now apply Menger's theorem to obtain a set $\mathcal{P}$ of k disjoint U-$N(v)$-paths. Extend each of those to a path to v in the obvious way, and we are done.