# SOLUTION FOR PROBLEM 3.7 IN CHVÁTAL'S BOOK

MARK DE LONGUEVILLE

**Theorem.** *Let $r_0, r_1, \dots, r_m$ and $s_0, s_1, \dots, s_m$ be two sequences of real numbers. Then there exists a $\delta > 0$ such that the following are equivalent:*

(i) $(r_0, r_1, \dots, r_m) < (s_0, s_1, \dots, s_m)$ *in the lexicographic order.*

(ii) *For every choice of numbers $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m$ such that $0 < \varepsilon_1 < \delta$ and $0 < \varepsilon_i < \varepsilon_{i-1}\delta$, $i = 2, 3, \dots, m$, the inequality*

$$r = r_0 + r_1\varepsilon_1 + \cdots + r_m\varepsilon_m < s = s_0 + s_1\varepsilon_1 + \cdots + s_m\varepsilon_m$$

*holds in the linear order of the reals.*

*Proof.* Assume the sequences $(r_0, r_1, \dots, r_m)$ and $(s_0, s_1, \dots, s_m)$ differ in at least one entry, since if they are equal the assertion is obviously correct. We can also assume that $m \geq 1$, since for $m = 0$ the assertion is obvious as well. Our assumptions allow us to define

$$c := \min_{j, r_j \neq s_j} |s_j - r_j|, \text{ and } C := \max_j |s_j - r_j|, \text{ and } \delta := \frac{c}{mC}.$$

Note that $0 < \delta \leq 1$. We now show the following: if $r_0 = s_0, r_1 = s_1, \dots, r_{k-1} = s_{k-1}$ and $r_k < s_k$ for some $k \in \{0, 1, \dots, m\}$ then $r < s$. Consider

$$s - r = (s_k - r_k)\varepsilon_k + (s_{k+1} - r_{k+1})\varepsilon_{k+1} + \cdots + (s_m - r_m)\varepsilon_m$$

for some choice of $\varepsilon_1, \dots, \varepsilon_m$ according to the restrictions stated above. This expression is positive because the following estimate holds:

$$\begin{aligned}
|(s_{k+1} - r_{k+1})\varepsilon_{k+1} + \cdots + (s_m - r_m)\varepsilon_m| &\leq |(s_{k+1} - r_{k+1})|\varepsilon_{k+1} + \cdots + |(s_m - r_m)|\varepsilon_m \\
&\leq C\varepsilon_{k+1} + \cdots + C\varepsilon_m \\
&\leq C(\varepsilon_{k+1} + \cdots + \varepsilon_m) \\
&\leq C\varepsilon_k\left(\frac{\varepsilon_{k+1}}{\varepsilon_k} + \cdots + \frac{\varepsilon_m}{\varepsilon_k}\right) \\
&< C\varepsilon_k(\delta + \delta^2 + \cdots + \delta^{m-k}) \\
&\leq C\varepsilon_k m\delta \\
&= C\varepsilon_k m\frac{c}{mC} \\
&= c\varepsilon_k \\
&\leq |(s_k - r_k)|\varepsilon_k.
\end{aligned}$$

This proves the equivalence of (i) and (ii) as follows. (i) implies (ii) since we just showed that if (i) holds then $r < s$ for any choice of $\varepsilon_1, \dots, \varepsilon_m$. If (i) does not hold then we must have that

$$(s_0, s_1, \dots, s_m) < (r_0, r_1, \dots, r_m).$$

Hence there is a $k \in \{0, 1, \ldots, m\}$ such that $r_0 = s_0, r_1 = s_1, \ldots, r_{k-1} = s_{k-1}$ and $s_k < r_k$. But then our proof above interchanging the roles of the $r_j$'s and the $s_j$'s just shows that $s < r$. $\square$

**Remark.** Observe that we actually proved something more general. We don't even need to consider a fixed pair of sequences $r_0, r_1, \ldots, r_m$ and $s_0, s_1, \ldots, s_m$. For fixed $0 < c \leq C$ we can consider any pair of sequences that satisfies

$$c \leq \min_{j, r_j \neq s_j} |s_j - r_j|, \text{ and } \max_j |s_j - r_j| \leq C$$

and our choice of $\delta$ works. In particular, this means that we can consider sequences of rational numbers with a fixed maximal encoding length, which suffices for all practical purposes. In our case we would define the encoding length as follows. Let $r = \frac{p}{q}$ be a rational number, where $p$ and $q$ are relative prime integers, then let the *encoding length* of $r$ be the sum of the number of digits of $p$ plus the number of digits of $q$. Why does this make sense? Because the following is true (why?). Let $r_0, r_1, \ldots, r_m$ and $s_0, s_1, \ldots, s_m$ be different sequences of rational numbers of encoding length at most $e$. Define $c$ and $C$ as above. Then

$$\frac{1}{2m10^{2e}} \leq \frac{c}{mC},$$

and therefore we can use $\delta = \frac{1}{2m10^{2e}}$ as a $\delta$ that works for arbitrary sequences with encoding length bounded by $e$.