Math 5711 Homework #7; due Wednesday, March 21, 2001

Mark de Longueville

Problem 1: Linear subspaces Let V be a linear subspace of $\mathbb{R}^n$. Show that there is an $m \geq 0$, and a matrix $A \in \mathbb{R}^{m\times n}$ such that

$$V = \{x \in \mathbb{R}^n : A \cdot x = 0\}.$$

What is the smallest possible m (depending on V, of course)?

Problem 2: Affine subspaces Let A be an affine subspace of $\mathbb{R}^n$, i.e., there exists a vector $a \in \mathbb{R}^n$ and a linear subspace V of $\mathbb{R}^n$ such that $A = a + V = \{a + x : x \in V\}$. Show that there is an $m \geq 0$, a vector $b \in \mathbb{R}^m$, and a matrix $A \in \mathbb{R}^{m\times n}$ such that

$$V = \{x \in \mathbb{R}^n : A \cdot x = b\}.$$

Problem 3: Affine spans

a) If $A = a + V$ is an affine linear subspace of $\mathbb{R}^n$ where $a \in A$ and $V \subseteq \mathbb{R}^n$ is a linear subspace, then $A = b + V$ for any $b \in A$.
b) Show that an affine linear space that contains the zero vector $0 \in \mathbb{R}^n$ is a linear subspace.
c) Let $S \subseteq \mathbb{R}^n$ be a set. Define the affine span $\mathrm{aff}(S)$ by

$$\mathrm{aff}(S) = \bigcap\{A : A \text{ affine subspace}, S \subseteq A\}.$$

Show that $\mathrm{aff}(S)$ is the (inclusionwise) smallest affine subspace that contains S.

Problem 4: Dimension We defined the dimension of a polyhedron P to be the dimension of the affine span of P, where the dimension of an affine space $a + V$ was defined to be the dimension of the linear space V.

Determine the dimensions of the following polyhedra according to our definition.

- $P_1 = \{x \in \mathbb{R}^2 : x_1 \leq 1\} \cap \{x \in \mathbb{R}^2 : x_2 \leq 3\} \cap \{x \in \mathbb{R}^2 : -x_1 \leq -1\} \cap \{x \in \mathbb{R}^2 : -x_2 \leq -3\}$
- $P_2 = \{x \in \mathbb{R}^2 : -x_1 \leq 0\} \cap \{x \in \mathbb{R}^2 : x_2 \leq 1\} \cap \{x \in \mathbb{R}^2 : x_2 - x_1 \leq -1\} \cap \{x \in \mathbb{R}^2 : x_1 - x_2 \leq 1\}$
- $P_3 = \{x \in \mathbb{R}^3 : x_1, x_2, x_3 \geq 0, x_1 + x_2 + x_3 \leq 3\}$
- $P_4 = \{x \in \mathbb{R}^3 : x_1, x_2, x_3 \geq 0, x_1 + x_2 + x_3 = 3\}$

Problem 5: Basic feasible solutions What are the basic feasible solutions of the following LP (compare the first midterm):

$$\begin{aligned} \max\ & 5x_1 + 4x_2 + 3x_3 \\ \text{subject to}\quad & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & -x_1 - 2x_2 - x_3 \leq -1 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Graph the polyhedron in question. Illustrate a path that the simplex algorithm possibly takes to the optimum.