Linear and combinatorial optimization - 5711
Midterm I
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Name:

#	Points
1	/25
2	/25
3	/25
4	/25
Sum	/100
Grade	

Problem 1 (25 points): Feasible dictionaries
Consider a linear program in standard form

$$(*) \qquad \max \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \qquad \sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad (i = 1, \dots, m),$$

$$x_j \ge 0 \qquad (j = 1, \dots, n),$$

and its first associated dictionary D given by:

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j \qquad (i = 1, \dots, m),$$

$$z = \sum_{j=1}^{n} c_j x_j.$$

Give the definition of a *feasible dictionary associated with the linear program* $(*)$, and introduce the notions of *basic* and *non-basic* variables. You might want to begin with

> "A feasible dictionary associated with the linear program $(*)$ is a system of linear equalities in the variables ... such that ..."

Problem 2 (25 points): Bicycle wheels

Velophil, Inc. makes spokes and rims for bicycle wheels. Recently, the firm has also started selling assembled wheels.

Each spoke requires 0.04 hours of machine time and 0.01 hours of labor, and costs \$0.05 to make. Each rim requires 0.6 hours of machine time and 0.5 hours of labor, at a cost of \$4.00.

An assembled wheel requires one rim, 36 spokes, and a hub that Velophil buys at a price of \$10 a piece. Assembling a wheel takes 1.2 hours of labor, in addition to that needed to make its rim and spokes; no further machine time is needed. The assembly process also incurs additional direct costs of \$11.00.

Up to 378 hours of machine time and 267 hours of labor are available per week.

Assembled wheels sell for \$60.00. Spokes not used to make assembled wheels can be sold for \$0.10 each. Similarly, rims not used in assembled wheels can be sold for \$10 each. Assume that, at these prices, the firm can sell all the products it can produce.

Algebraically formulate a linear program to maximize Velophil's weekly profits. Give clear, numeric definitions of your decision variables.

Problem 3 (25 points): The two-phase simplex algorithm

Solve the following linear program with the simplex method.

$$
\begin{aligned}
& \max\ 5x_1 + 4x_2 + 3x_3 \\
\text{subject to} \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\
& 3x_1 + 4x_2 + 2x_3 \leq 8 \\
& -x_1 - 2x_2 - x_3 \leq -1 \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}
$$

Problem 4 (25 points): 2-dimensional problems
Consider a 2-dimensional linear program in standard form

$$
(*) \qquad \begin{aligned} & \max c_1x_1 + c_2x_2 \\ \text{subject to} \quad & a_{i1}x_1 + a_{i2}x_2 \le b_i \qquad (i = 1, \ldots, m), \\ & x_1, x_2 \ge 0. \end{aligned}
$$

Prove that

(i) (x_1, x_2) is a basic feasible solution of $(*)$

implies

(ii) (x_1, x_2) is a feasible solution of $(*)$ that is defined by the intersection of two of the lines

$$
\begin{aligned} l_1 : & \quad x_1 = 0, \\ l_2 : & \quad x_2 = 0, \\ l_{2+i} : & \quad a_{i1}x_1 + a_{i2}x_2 = b_i \qquad (i = 1, \ldots, m). \end{aligned}
$$

Remark: The implication in the other direction also holds. You might want to think about it after the exam.